On power subgroups of Dehn twists in hyperelliptic mapping class groups

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1. Power subgroups of mapping class groups
2. Main results
3. The Kauffman bracket skein module
4. Quantum representations of $\mathcal{M}(0, 2n)$
5. Infiniteness of the index of a power subgroup of $\mathcal{M}(0, 2n)$

Mapping class groups and power subgroups

Let $\mathcal{M}(g,p)$ be the mapping class group of an oriented connected compact surface Σ_g^p of genus g with p punctures.

- **Remark** Let c be a simple closed curve (SCC).

  t_c a Dehn twist along c,

  σ_i a half twist exchanging the i-th and $i+1$-th marked points.

We consider "power subgroups" of $\mathcal{M}(g,p)$. Examples of power subgroups are the following.

Let c be a non-separating simple closed curve (SCC) on Σ_g^p and $m \in \mathbb{Z}_{\geq 0}$.

$\mathcal{N}_m(g,p)$ the normal closure of t_c^m in $\mathcal{M}(g,p)$,

$\tilde{\mathcal{N}}_m(g,p)$ the normal closure of m-th powers of all Dehn twists in $\mathcal{M}(g,p)$,

$N_m(g,p)$ the normal closure of $\{\sigma_i^m \mid i = 1, 2, \ldots p-1\}$ in $\mathcal{M}(g,p)$.

Question

Is the indices of a power subgroup FINITE or INFINITE?

Indices of power subgroups

- **In the case of $\mathcal{N}_m, \tilde{\mathcal{N}}_m \subset \mathcal{M}(g,0)$:**
  - **Theorem (Newman 1972)** $\left[\mathcal{M}(1,0); \tilde{\mathcal{N}}_m\right] = \infty$ if $m \geq 6$, and finite if $m < 6$.
  - **Theorem (Humphries 1992)** $\left[\mathcal{M}(2,0); \tilde{\mathcal{N}}_m\right] = \infty$ if $m \geq 4$, and finite if $m < 4$.
  - **Theorem (Funar 1999)** $[\mathcal{M}(g,0); \mathcal{N}_m] = \infty$ if $g \geq 3$ and $m \notin \{1,2,3,4,6,8,12\}$.

- **In the case of $N_m \subset \mathcal{M}(0,2n)$:**
  - **Theorem (Stylianakis 2018)** $[\mathcal{M}(0,2n); N_m] = \infty$ if $2n \geq 6$ and $m \geq 5$.
    - Stylianakis used the Jones representation at root of unity.
  - **Theorem (Masbaum 2018)** $[\mathcal{M}(0,2n); N_m] = \infty$ if $2n \geq 4$ and $m \geq 6$.
    - Masbaum used the quantum representation obtained from the Kauffman bracket skein module.

Main results

Masbaum's comments

"I believe that the remaining case ($2n \geq 6$, $m = 5$) can also be proved by using the skein theory and the proof requires some mathematical software."

Theorem (Y.)

$[\mathcal{M}(0, 2n); N_m] = \infty$ if $2n \geq 6$ and $m = 5$, $m \geq 7$. The proof used the skein theory and hand calcululation.

Let Δ_g be the hyperelliptic mapping class group of Σ_g^0 equipped with a hyperelliptic involution ι. We define power subgroups of Δ_g as follows:

- $\mathcal{N}_m^\iota$ the normal closure of the m-th power of a Dehn twist along symmetric non-separating SCC,
- $\tilde{\mathcal{N}}_m^\iota$ the normal subgroup of m-th powers of Dehn twists along all symmetric SCCs.

Corollary (Y.) $\left[\Delta_g; \tilde{\mathcal{N}}_m^\iota\right] = \infty$ if $g \geq 2$ and $m \geq 4$.

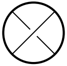
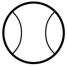
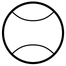
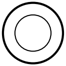
Remark Stylianakis showed $[\Delta_g; \mathcal{N}_m^\iota] = \infty$ if $g \geq 2$ and $m \geq 4$ as a corollary of his theorem.

The Kauffman bracket skein relation

Let us describe a part of framed tangle in an oriented 3-manifold M as a tangle diagram in a disk. The framing is given by a blackboard framing.

Definition (The Kauffman bracket skein relation)

Let q be an invertible elements in $\mathbb{C}$. The Kauffman bracket skein relations is relations in the $\mathbb{C}$-vector space spanned by framed tangles in M defined as follows:

-  $= q^{\frac{1}{4}}$  $+ q^{-\frac{1}{4}}$ ,
- $L \sqcup$  $= -[2]\,L,$ for any tangle L.

The definition of the quantum integer $[n]$ is

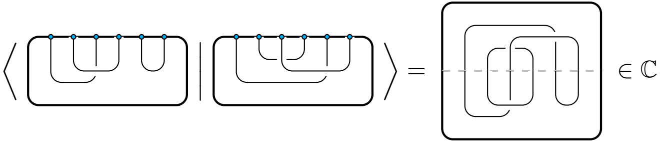
$$[n] = (q^{\frac{n}{2}} - q^{-\frac{n}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$$

The Kauffman bracket skein modules

Let $([0,1]^3, 2n)$ be the 3-ball with $2n$ marked points on the top side. The Kauffman bracket skein module of $([0,1]^3, 2n)$ is

$$\mathrm{S}_q(1^{\otimes 2n}) = \mathrm{span}_{\mathbb{C}}\{\text{tangles in } ([0,1]^3, 2n)\}/\text{the Kauffman bracket skein relations.}$$

There is a natural Hermitian form $\langle \cdot \mid \cdot \rangle \colon \mathrm{S}_q(1^{\otimes 2n}) \times \mathrm{S}_q(1^{\otimes 2n}) \to \mathbb{C}$ defined by gluing two $([0,1]^3, 2n)$'s together at top sides. The latter cube takes the mirror image.



We consider the quatient vector space $\mathcal{S}_q(1^{\otimes 2n})$ of $\mathrm{S}_q(1^{\otimes 2n})$ which makes $\langle \cdot \mid \cdot \rangle$ non-degenerate.

Bases of $\mathcal{S}_q(1^{\otimes 2n})$

!!! In the follwoing, we take q as a primitive r-th root of unity !!!

A basis of $\mathcal{S}_q(1^{\otimes 2n})$ is given by uni-trivalent graph with admissible colorings.

Definition (r-admissible coloring)

Let T be a uni-trivalent graph whose edges are labelled by non-negative integers (we call them colors). A triple of colors (a, b, c) on edges adjacent to a trivalent vertex v is r-admissible if

- $a + b + c$ is even,
- $a + b - c$, $b + c - a$, $c + a - b$ are non-negative,
- $0 \leq a, b, c \leq r - 2$ and $a + b + c \leq 2(r - 2)$.

An r-admissible coloring on T is a colorings whose triples of colors are r-admissible for any trivalent vertices.

A basis of $\mathcal{S}_q(1^{\otimes 2n})$ is given by the set of r-admissible colorings on an uni-trivalent graph in a disk with $2n$ marked points. The graph is considered as an embedded graph in the vertical plane $\{1/2\} \times [0,1]^2 \subset ([0,1]^3, 2n)$.

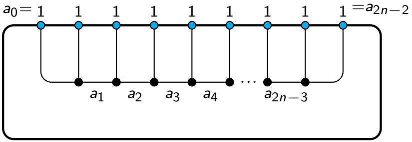
Bases of $\mathcal{S}_q(1^{\otimes 2n})$

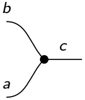
Theorem (Lickorish 1997 etc.)

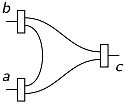
Fix an embedding of uni-trivalent graph T into the disk with $2n$ marked points. Then, the basis of $\mathcal{S}_q(1^{\otimes 2n})$ is obtained by the set of r-admissible colorings on T as the following example.

The trivalent graph T with r-admissible colorings $(a_1, a_2, \ldots, a_{2n-3})$ gives a basis of $\mathcal{S}_q(1^{\otimes 2n})$

$$\beta_T(a_1, a_2, \ldots, a_{2n-3}) =$$



where 

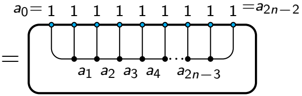
represents a skein element 

, and white boxes represent “Jones-Wenzl projectors”.

Coodinate changes and 6j-symbol

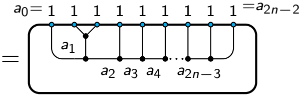
Let T and T' are embedded uni-trivalent graphs in the disk with $2n$ marked points. Then, one can deform T into T' by a sequence of "flips".
For example, the underlying uni-trivalent graph T and T' of

$$\beta_T(a_1, a_2, \ldots, a_{2n-3}) =$$



and

$$\beta_{T'}(a_1, a_2, \ldots, a_{2n-3}) =$$



,

are related by a flip on an edge (colored with a_1).
A flip of uni-trivalent graph at some edge induce a coodinate change of $\mathcal{S}_q(1^{\otimes 2n})$.
We denote these orderd bases by

- $\mathcal{B}_T = \{\, \beta_T(a_1, a_2, \ldots, a_{2n-3}) \mid r\text{-admissible} \,\}$,
- $\mathcal{B}_{T'} = \{\, \beta_{T'}(a_1, a_2, \ldots, a_{2n-3}) \mid r\text{-admissible} \,\}$.

The order is the lexicographic order of colors $(a_1, a_2, \ldots, a_{2n-3})$.

Coodinate changes and $6j$-symbol

Remark The r-admissibility requires the following conditions:

$$\begin{aligned} &\text{Type I} && (a_1, a_2, \dots, a_{2n-3}) \text{ such that } a_1 = 0 \text{ and } a_2 = 1, \\ &\text{Type II} && (a_1, a_2, \dots, a_{2n-3}) \text{ such that } a_1 = 2 \text{ and } a_2 = 1, \\ &\text{Type III} && (a_1, a_2, \dots, a_{2n-3}) \text{ such that } a_1 = 2 \text{ and } a_2 = 3. \end{aligned}$$

$\mathcal{B}_T$ transforms into $\mathcal{B}_{T'}$ by $6j$-symbols,

$$\beta_T(0,1,a_3,\dots,a_{2n-3}) = \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix} \beta_{T'}(0,1,a_3,\dots,a_{2n-3}) + \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} \beta_{T'}(2,0,a_3,\dots,a_{2n-3})$$

$$\beta_T(2,1,a_3,\dots,a_{2n-3}) = \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix} \beta_{T'}(2,1,a_3,\dots,a_{2n-3}) + \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix} \beta_{T'}(0,1,a_3,\dots,a_{2n-3})$$

$$\beta_T(2,3,a_3,\dots,a_{2n-3}) = \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \end{Bmatrix} \beta_{T'}(2,3,a_3,\dots,a_{2n-3}).$$

The value of the above $6j$-symbols are the following:

$$\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix} = -\frac{1}{[2]}, \quad \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} = 1, \quad \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix} = \frac{1}{[2]},$$

$$\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{Bmatrix} = \frac{[3]}{[2]^2}, \quad \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{Bmatrix} = 1.$$

Consequently, the coodinate change matrix is $\begin{bmatrix} -[2]^{-1} I_k & [3][2]^{-2} I_k & O \\ I_k & [2]^{-1} I_k & O \\ O & O & I_{k'} \end{bmatrix}$.

Braid groups and Mapping class groups of punctured spheres

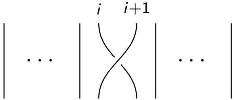
Definition (Braid groups)

The braid group of k strands has a presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{k-1} \,\middle|\, \begin{matrix} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (1 \leq i < k-1) \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| > 1) \end{matrix} \right\rangle.$$

The generator σ_i represents the following braid diagram:

$\sigma_i =$



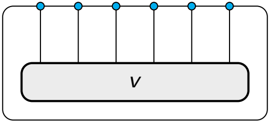
for $i = 1, 2, \ldots, k-1$.

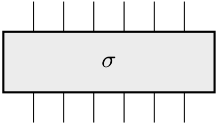
The composition of two braid diagrams ab is given by gluing b on the top of a.

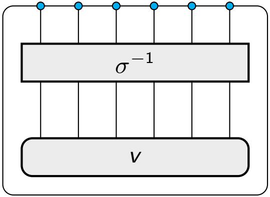
We construct a quantum representation of the mapping class group of punctured sphere $\mathcal{M}(0, 2n)$ from an action of B_{2n} on $\mathcal{S}_q(1^{\otimes 2n})$.
The action of $\sigma \in B_{2n}$ on $\mathcal{S}_q(1^{\otimes 2n})$ is defined by gluing σ^{-1} on the top of $([0,1]^3, 2n)$.

The action of B_{2n} on $\mathcal{S}_q(1^{\otimes 2n})$

The action $\tilde{\rho}$ of B_{2n} on the clasped web spaces $\mathcal{S}_q(1^{\otimes 2n})$ is defined as follows.

$\mathcal{S}_q(1^{\otimes 6}) \ni$ 

$B_6 \ni$ 

$\tilde{\rho}(\sigma)(v) =$ 

Quantum represenatations of $\mathcal{M}(0, 2n)$

We construct a quantum representation of $\mathcal{M}(0, 2n)$ from $\tilde{\rho}$. It is well-known that $\mathcal{M}(0, 2n)$ is a quotient of B_{2n} by two relators:

$$R_1 = \sigma_1\sigma_2\dots\sigma_{2n-1}\sigma_{2n-1}\sigma_{2n-2}\dots\sigma_1$$
$$R_2 = (\sigma_1\sigma_2\dots\sigma_{2n-1})^{2n}.$$

One can obtain $\tilde{\rho}(R_1) = \tilde{\rho}(R_2) = \mathrm{Id}$ in $PGL(\mathcal{S}_q(1^{\otimes 2n}))$ by easy calculations. Thus, the projectivization of $\tilde{\rho}\colon B_{2n} \to GL(\mathcal{S}_q(1^{\otimes 2n}))$ factors through $\rho\colon \mathcal{M}(0, 2n) \to PGL(\mathcal{S}_q(1^{\otimes 2n}))$.

$$\begin{array}{ccc} B_{2n} & \xrightarrow{\quad\tilde{\rho}\quad} & GL(\mathcal{S}_q(1^{\otimes 2n})) \\ \Big\downarrow{\scriptstyle /R_1, R_2} & & \Big\downarrow{\scriptstyle \text{projectivization}} \\ \mathcal{M}(0, 2n) & \xrightarrow{\quad\rho\quad} & PGL(\mathcal{S}_q(1^{\otimes 2n})) \end{array}$$

Strategy for proving inifiniteness of the index of a power subgroup

Masbaum's strategy

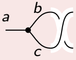
For given power m, find a primitive r-th root of unity q and $f \in \mathcal{M}(0, 2n)$ such that

1. $\rho(\sigma_1^m) = \mathrm{Id}$,
2. $\rho(f)$ has inifinite order in $PGL(\mathcal{S}_q(1^{\otimes 2n}))$.

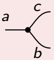
Remark σ_1 is conjugate to σ_i for any $i = 1, 2, \ldots, 2n - 1$.

For any embedded uni-trivalent graph T in $([0,1]^3, 2n)$, ρ_T denotes the matrix representation of ρ with respect to the ordered basis $\mathcal{B}_T$.

- **Strategy 1 (easy)** Computing $\rho(\sigma_1^m)$ by using the basis $\mathcal{B}_T$ and a twist formula



$$= (-1)^{\frac{a-b-c}{2}} q^{-\frac{1}{8}(a(a+2)-b(b+2)-c(c+2))}$$



$$\rightsquigarrow \quad \rho(\sigma_1^m)(\beta_T(0, a_1, \ldots, a_{2n-3})) = (-1)^m q^{\frac{3m}{4}} \beta_T(0, a_1, \ldots, a_{2n-3}),$$

$$\rho(\sigma_1^m)(\beta_T(2, a_1, \ldots, a_{2n-3})) = q^{-\frac{m}{4}} \beta_T(2, a_1, \ldots, a_{2n-3}).$$

Strategy for proving inifiniteness of the index of a power subgroup

Masbaum's strategy

For given power m, find a primitive r-th root of unity q and $f \in \mathcal{M}(0, 2n)$ such that

1. $\rho(\sigma_1^m) = \mathrm{Id}$,
2. $\rho(f)$ has inifinite order in $PGL(\mathcal{S}_q(1^{\otimes 2n}))$.

Solution for strategy 1

- If m is even, then q is a m-th root of unity,
- If m is odd, then q is a $2m$-th root of unity.

- **Strategy 2** Suppose $\rho(f)^N = \mathrm{Id}$ in $PGL(\mathcal{S}_q(1^{\otimes 2n}))$, then the lift $\sigma \in B_{2n}$ of f satisfies $\tilde{\rho}(\sigma^N) = \lambda \mathrm{Id}$ in $GL(\mathcal{S}_q(1^{\otimes 2n}))$ for some $\lambda \in \mathbb{C}$. This condition implies

$$|\mathrm{trace}(\tilde{\rho}(\sigma))| \leq \mathrm{rank}(\tilde{\rho}(\sigma))$$

■ Find $\sigma \in B_{2n}$ satisfying $|\mathrm{trace}(\tilde{\rho}(\sigma))| > \mathrm{rank}(\tilde{\rho}(\sigma)) \cdots (\clubsuit)$.

Review of Masbaum's calculation for the remaining case ($m = 5$)

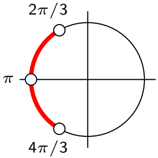
Masbaum compute $\tilde{\rho}_T(\sigma)$ where $\sigma = \sigma_1^2\sigma_2^{-2} \in B_{2n}$. One can compute

$$\rho_T(\sigma) = \begin{bmatrix} (1+q^2[3])[2]^{-2} I_k & (-1+q^2)[3][2]^{-3} I_k & O \\ (1-q^{-2})[2]^{-1} I_k & (1+(-1)^s q^{-2}[3])[2]^{-2} I_k & O \\ O & O & I_{k'} \end{bmatrix}.$$

$$\mathrm{tr}(\rho_T(\sigma)) = f(q)k + k' \text{ where } f(q) = (q^2+q^{-2}) - \left(\frac{q-q^{-1}}{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}\right)^2.$$

Set $q = \exp(i\theta)$, the condition (♣) is satistied when

$$\frac{2\pi}{3} < \theta < \pi, \pi < \theta < \frac{4\pi}{3}.$$



In the case of $m = 5$, the strategy 1 requires q of a primitive 10-th root of unity. However, $\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5} \cdots$ ● do not satisfy the condition.

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$$\rho_T(\sigma) = \begin{bmatrix} (1 + q^2\,[3])\,[2]^{-2}\,I_k & (-1 + q^2)\,[3]\,[2]^{-3}\,I_k & O \\ (1 - q^{-2})\,[2]^{-1}\,I_k & (1 + (-1)^s q^{-2}\,[3])\,[2]^{-2}\,I_k & O \\ O & O & I_{k'} \end{bmatrix}.$$

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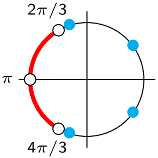

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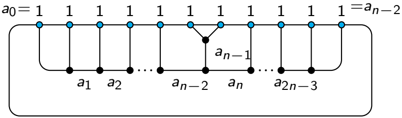
Our calculation for the remaining case ($m = 5$)

We compute $\tilde{\rho}_T(\sigma)$ where

$$\sigma = (\sigma_1\sigma_2 \dots \sigma_{n-1})^n \sigma_n (\sigma_1\sigma_2 \dots \sigma_{n-1})^{-n} \sigma_n^{-1} \in B_{2n}.$$

To calculate $\tilde{\rho}_T(\sigma)$, we use the ordered basis $\mathcal{B}_T$ and new basis $\mathcal{B}_Y$ whose elements are

$\beta_Y(a_1, a_2, \dots, a_{2n-3}) =$ 

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One can compute $\tilde{\rho}_T(\sigma)$ as follows:

$$(\mathcal{B}_T \overset{6j}{\rightsquigarrow} \mathcal{B}_Y) \to \tilde{\rho}_Y(\sigma_n^{-1}) \to (\mathcal{B}_Y \overset{6j}{\rightsquigarrow} \mathcal{B}_T) \to \tilde{\rho}_T((\sigma_1\sigma_2 \dots \sigma_{n-1})^{-n})$$
$$\to (\mathcal{B}_T \overset{6j}{\rightsquigarrow} \mathcal{B}_Y) \to \tilde{\rho}_Y(\sigma_n) \to (\mathcal{B}_Y \overset{6j}{\rightsquigarrow} \mathcal{B}_T) \to \tilde{\rho}_T((\sigma_1\sigma_2 \dots \sigma_{n-1})^{n})$$

We use the twist formula for $\tilde{\rho}_Y(\sigma_n)$ and $\tilde{\rho}_T((\sigma_1\sigma_2 \dots \sigma_{n-1})^n)$ makes a curl on a edge of T colored with a_{n-1}.

Our calculation for the remaining case ($m = 5$)

We compute $\rho_T(\sigma)$ for odd $n \geq 3$, and for even $n \geq 4$. We only see the case of odd $n \geq 3$.
The direct computation of the matrix representation $M = \tilde{\rho}_T(\sigma)$ gives

$$M = \begin{cases} I_{k_{r-4}} \oplus I_{l_1(1)} \oplus M(1) \oplus M(3) \oplus \cdots \oplus M(r-4) & \text{if } r \text{ is odd,} \\ I_{k_{r-3}} \oplus I_{l_1(1)} \oplus M(1) \oplus M(3) \oplus \cdots \oplus M(r-5) \oplus I_{l_2(r-3)} & \text{if } r \text{ is even,} \end{cases}$$

where

$$M(a) = \begin{bmatrix} \left(1 - (1 - f_a(q) f_{a+2}(q^{-1})) \frac{[a+1][a+3]}{[a+2]^2}\right) I_{l_2(a)} & (1 - f_a(q) f_{a+2}(q^{-1})) \frac{(-q^{\frac{1}{2}}[a+1] + q^{-\frac{1}{2}}[a+3])[a+1][a+3]}{[2][a+2]^3} I_{l_2(a)} \\ (1 - f_a(q^{-1}) f_{a+2}(q)) \frac{q^{-\frac{1}{2}}[a+1] - q^{\frac{1}{2}}[a+3]}{[2][a+2]} I_{l_2(a)} & \left(1 - (1 - f_a(q^{-1}) f_{a+2}(q)) \frac{[a+1][a+3]}{[a+2]^2}\right) I_{l_2(a)} \end{bmatrix},$$

and $f_a(q) = (-1)^a (q^a + q^{-a}) + (-1)^{a+1} \left(\frac{q^{\frac{a}{2}} + (-1)^{a+1} q^{-\frac{a}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \right)^2.$

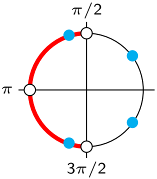
Thus,

$$\begin{aligned} \operatorname{tr}(M(a)) &= \left(2 - (2 - f_a(q) f_{a+2}(q^{-1}) - f_a(q^{-1}) f_{a+2}(q)) \frac{[a+1]\,[a+3]}{[a+2]^2}\right) l_2(a) \\ &= 2 l_2(a) + (q^{\frac{a+1}{2}} - q^{-\frac{a+1}{2}})(q^{\frac{a+3}{2}} - q^{-\frac{a+3}{2}}) l_2(a) \end{aligned}$$

Our calculation for the remaining case ($m = 5$)

The condition (♣) is $(q^{\frac{a+1}{2}} - q^{-\frac{a+1}{2}})(q^{\frac{a+3}{2}} - q^{-\frac{a+3}{2}}) > 0$. Then, the angle of $q = \exp(i\theta)$ is

$$\frac{\pi}{2} < \theta < \pi, \pi < \theta < \frac{3\pi}{2}.$$



In fact, 10-th roots of unity $q = \exp(3\pi/5), \exp(7\pi/5)$ satisfy this condition. Such direct computations of $\tilde{\rho}(\sigma)$ show the following:

Theorem (Y.)

For any $2n \geq 6$, $m \geq 5$ and $m \neq 6$, N_m has infinite index in $\mathcal{M}(0, 2n)$.

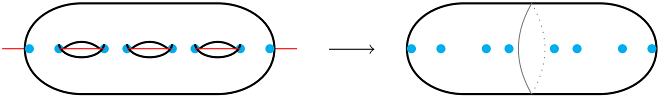
Power subgroups of hyperelliptic mapping class groups

Let $\Sigma_g = \Sigma_g^0$ is an oriented closed surface of genus g equipped with a hyperelliptic involution $\iota\colon \Sigma_g \to \Sigma_g$.
The hyperelliptic mapping class group Δ_g is the centralizer of the isotopy class of ι in $\mathcal{M}(g, 0)$.

$$\Delta_g = \{f \in \mathcal{M}(g, 0) \mid f\iota = \iota f\}$$

Remark For a SCC α on Σ_g, $t_\alpha \in \Delta_g$ if $\iota(\alpha) = \alpha$.
The follwong theorem relates Δ_g to $\mathcal{M}(0, 2g+2)$:

Theorem (Birman-Hilden 1973) $\Delta_g/\langle\iota\rangle \cong \mathcal{M}(0, 2g+2)$.



Power subgroups of hyperelliptic mapping class groups

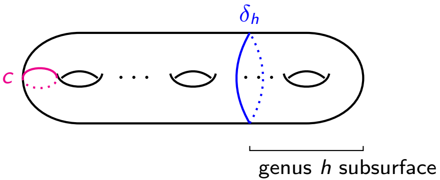
A quantum representaiton

$$\rho' \colon \Delta_g \to PGL(\mathcal{S}_q(1^{\otimes 2g+2}))$$

is defined as a composition of the surjective homomorphism $\Delta_g \to \mathcal{M}(0, 2g+2)$ and $\rho \colon \mathcal{M}(0, 2g+2) \to PGL(\mathcal{S}_q(1^{\otimes 2g+2}))$.

We study a power sugbroup $\mathcal{N}^{\iota}_{(k,\ell)}$ of Δ_g through $\rho' \colon \Delta_g \to PGL(\mathcal{S}_q(1^{\otimes 2g+2}))$.

$$\mathcal{N}^{\iota}_{(k,\ell)} = \text{the normal closure of } \{t_c^k, t_{\delta_h}^{\ell} \mid h = 1, 2, \dots g-1\} \text{ in } \Delta_g$$



We use the same strategy to Masbaum's one:

Strategy for Δ_g

For given powers k, ℓ, find a primitive r-th root of unity q and $f \in \Delta_g$ such that

1. $\rho'(t_c^k) = \mathrm{Id}$, and $\rho'(t_{\delta_h}^\ell) = \mathrm{Id}$ for any $h = 1, 2, \ldots, g-1$,
2. $\rho'(f)$ has inifinite order in $PGL(\mathcal{S}_q(1^{\otimes 2n}))$.

Remark

The projection of $t_c \in \Delta_g$ on $\mathcal{M}(0, 2g+2)$ is σ_1, and t_{δ_h} is $(\sigma_1\sigma_2\ldots\sigma_{2h})^{4h+2}$

We only have to calculate $\rho'(t_{\delta_h})$ and and solve the strategy 1.

Proposition (Y.)

Let q be a primitive r-th root of unity and $\rho\colon \Delta_g \to \mathcal{S}_q(1^{\otimes 2g+2})$ the projective representation. Then,

1. $\rho(t_{\delta_h}^\ell) = \mathrm{Id}$ for any $g \geq 2$ and $1 \leq h \leq \lfloor g/2 \rfloor$ when $r = 4$,
2. $\rho(t_{\delta_h}^\ell) = \mathrm{Id}$ if $q^{6\ell} = 1$ for any $g \geq 2$ and $1 \leq h \leq \lfloor g/2 \rfloor$ when $r = 5, 6$,
3. $\rho(t_{\delta_1}^\ell) = \mathrm{Id}$ if $q^{6\ell} = 1$ for $g = 2, 3$ when $r \geq 7$.
4. $\rho(t_{\delta_h}^\ell) = \mathrm{Id}$ if $q^{2\ell} = 1$ for any $g \geq 4$ and $1 \leq h \leq \lfloor g/2 \rfloor$ when $r \geq 7$.

We use a lift of $\sigma = (\sigma_1\sigma_2 \ldots \sigma_{n-1})^n \sigma_n (\sigma_1\sigma_2 \ldots \sigma_{n-1})^{-n} \sigma_n^{-1}$ as f in the strategy 2. The computation of $\rho'(f)$ is similar to $\rho(\sigma)$, and we already computeted the matrix.
Consequently, we obtain the follwoing:

Theorem (Y.)

1. $g = 2, 3,\ m \geq 1,\ \mathcal{N}^{\iota}_{(6m+3,2m+1)}$ and $\mathcal{N}^{\iota}_{(6m+9,2m+3)}$ have infinite indices in Δ_g.
2. $g = 2, 3,\ m \geq 2,\ \mathcal{N}^{\iota}_{(6m,m)}$ and $\mathcal{N}^{\iota}_{(6m+6,m+1)}$ have infinite indices in Δ_g.
3. $g \geq 4,\ m \geq 2,\ \mathcal{N}^{\iota}_{(2m,m)}$ and $\mathcal{N}^{\iota}_{(2m+1,2m+1)}$ have infinite indices in Δ_g.

Corollary (Y.) $\left[\Delta_g; \tilde{\mathcal{N}}^{\iota}_m\right] = \infty$ if $g \geq 2$ and $m \geq 4$.

$\tilde{\mathcal{N}}^{\iota}_m = \mathcal{N}^{\iota}_{(m,m)}$ and the third case show the corollary.