Tunnel number of knots and generalized tangles

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Previous results

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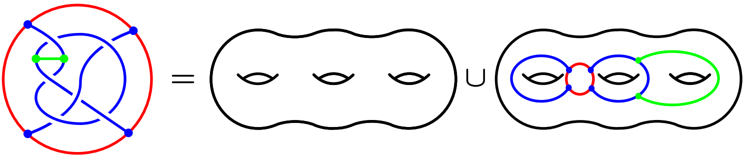
Let (B, T) be an n-tangle. A disjoint union τ of simple arcs joining T to itself is called an *unknotting tunnel system* if $\mathrm{Ext}(\partial B \cup T \cup \tau; B)$ is a handlebody. The minimal number of such arcs is called the *tunnel number* $\mathrm{tnl}(T)$ of (B, T).



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Let K be a knot in S^3 and $T_1 \cup T_2$ an n-tangle decomposition of K. Then $\mathbf{tnl}(K) \leq \mathbf{tnl}(T_1) + \mathbf{tnl}(T_2) + 2n - 1$.

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Remark.

If $\mathrm{tnl}(T_1) = \mathrm{tnl}(T_2) = 0$ (i.e. both T_1 and T_2 are free), this corresponds to Morimoto's conjecture.

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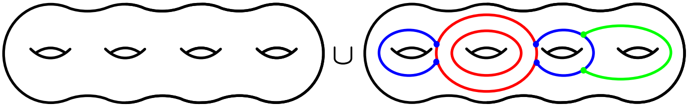

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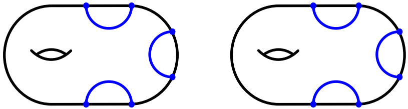

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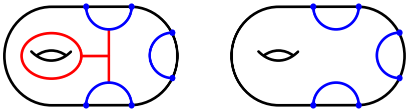

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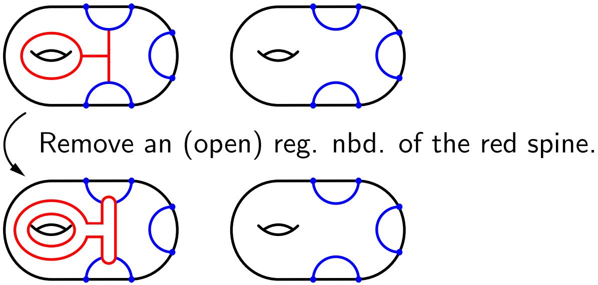

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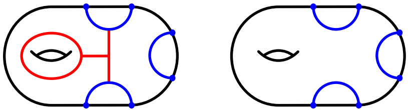
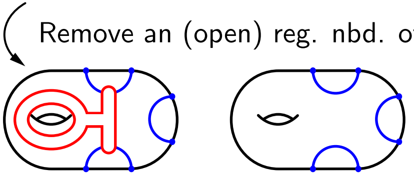


Remove an (open) reg. nbd. of the red spine.



$V := M^3 - \mathbf{Nbd(Red\ Spine)}$,

$T := K \cap V$.

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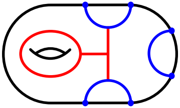
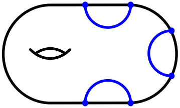
$(S^3, K) =$  $\cup$ 

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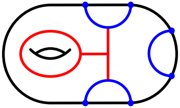
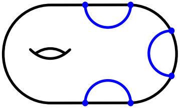
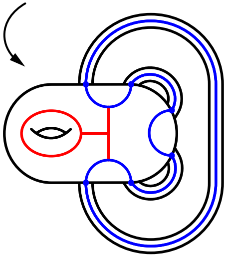
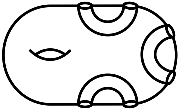
$(S^3, K) =$  $\cup$ 

$=$  $\cup$ 

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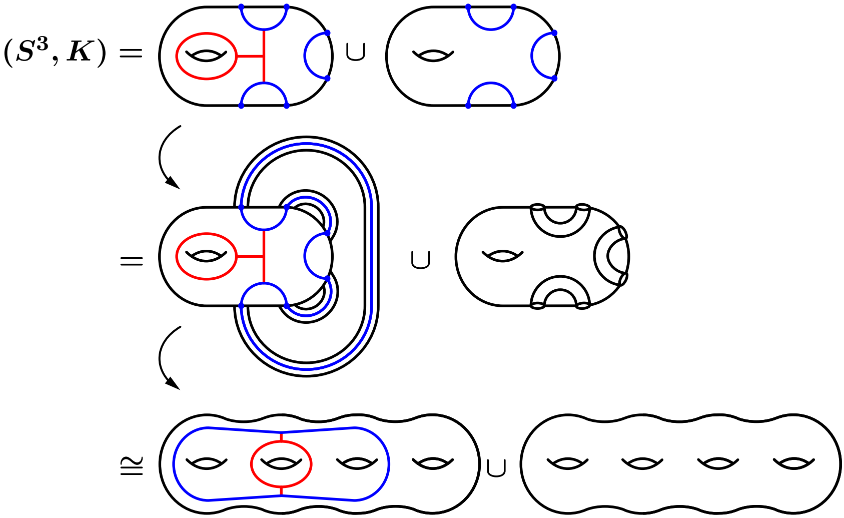

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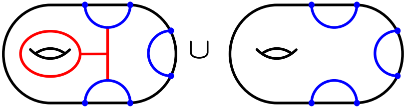
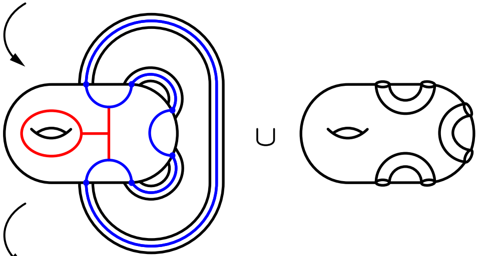
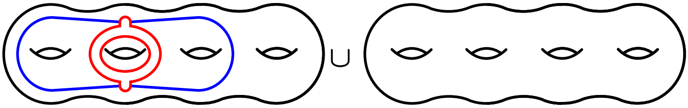
$(S^3, K) =$ 

$=$ 

$\cong$ 

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In general, we can obtain a (g, n)-tangle (V, T) of tunnel number t from a $(g, n + t)$-knot of tunnel number $g + n + t - 1$.

How to make a (g, n)-tangle (V, T) of tunnel number t

In general, we can obtain a (g, n)-tangle (V, T) of tunnel number t from a $(g, n + t)$-knot of tunnel number $g + n + t - 1$.

Remark (Ichihara-S., 2013).

Such a knot does exist.

c.f.) Knots with arbitrarily high distance bridge decompositions, Bull. Korean Math. Soc. 50 (2013), no. 6, 1989–2000.

Results

Theorem 1.

Let K be a knot in a closed orientable 3-manifold and $T_1 \cup T_2$ a (g, n)-tangle decomposition of K. Then

$$\mathrm{tnl}(K) \leq \mathrm{tnl}(T_1) + \mathrm{tnl}(T_2) + g + 2n - 1.$$

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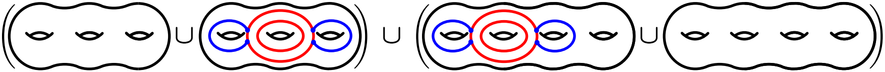

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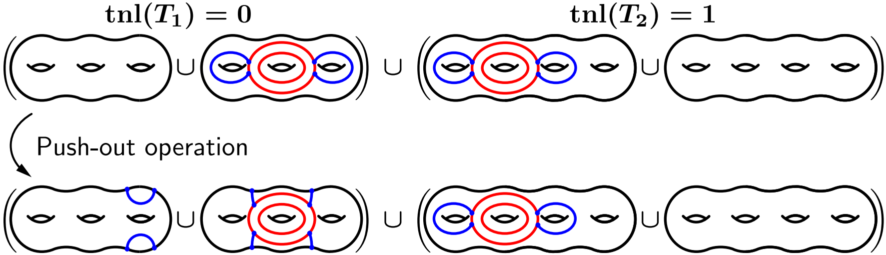

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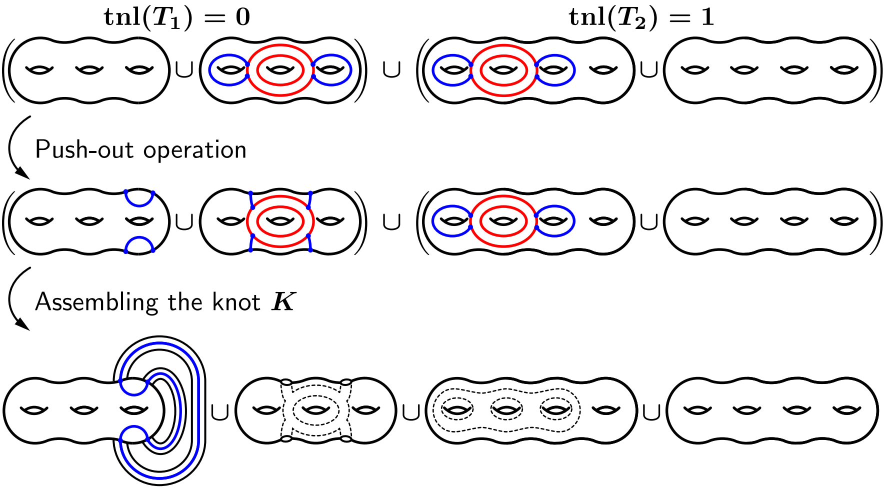

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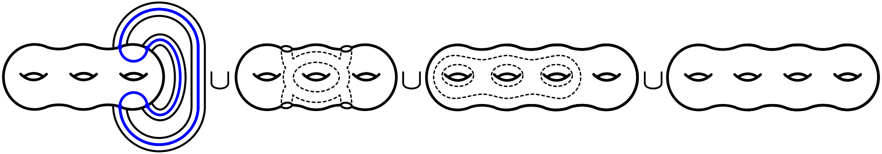

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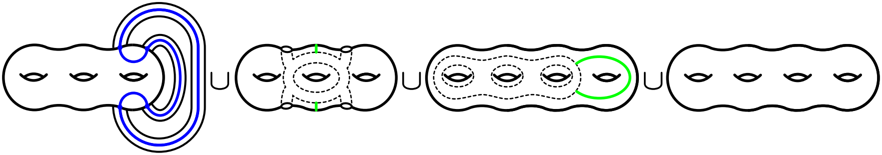

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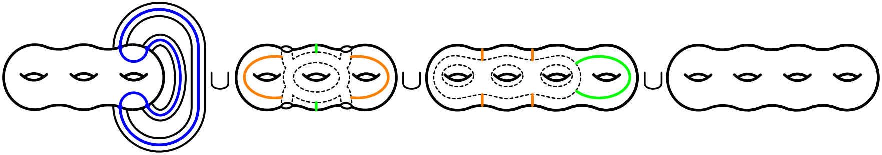

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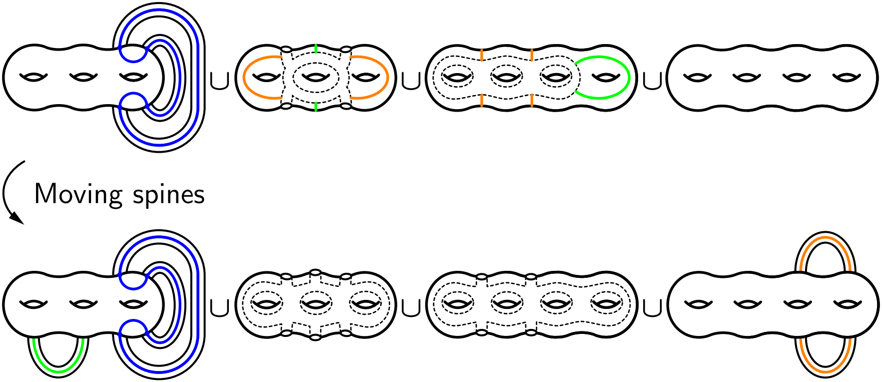

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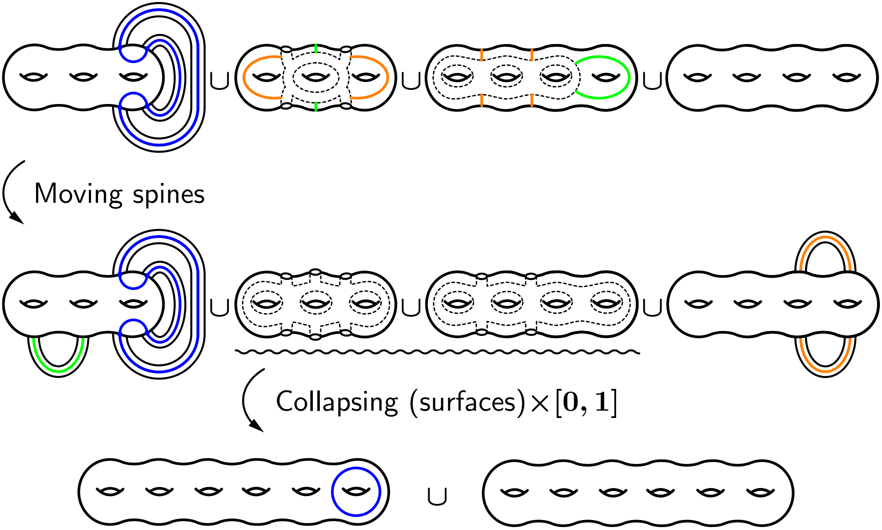

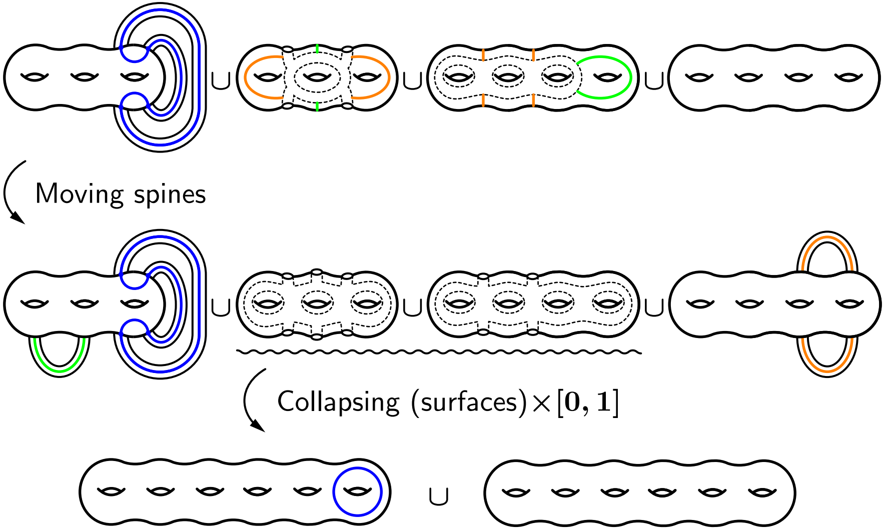

$$\mathbf{tnl}(K) \leq \mathbf{5}.$$

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Question.

$\forall g \in \mathbb{Z}_{\geq 1}$, $\forall n \in \mathbb{Z}_{\geq 2}$, $\forall t_1, \forall t_2 \in \mathbb{Z}_{\geq 0}$, $\exists ? K = T_1 \cup T_2$: a (g, n)-tangle decomposition s.t. $\mathrm{tnl}(T_1) = t_1$, $\mathrm{tnl}(T_2) = t_2$ and

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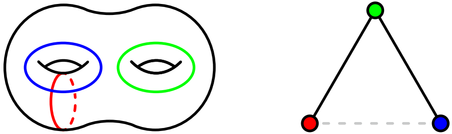
a very complicated tangle ≈ a high distance tangle

Curve complex

Definition.

Let S be a closed orientable surface. The *curve complex* of S, denoted by $\mathcal{C}(S)$, is the complex such that:

- the vertices are the isotopy classes of essential simple loops in S, and
- distinct k vertices determine a k-simplex if they correspond to pairwise disjoint loops.



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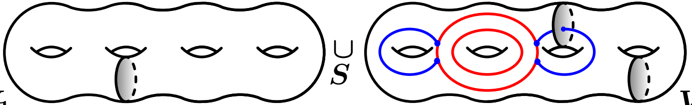

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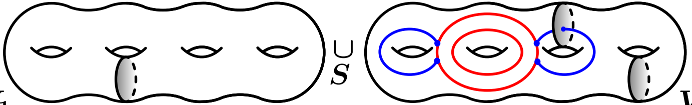
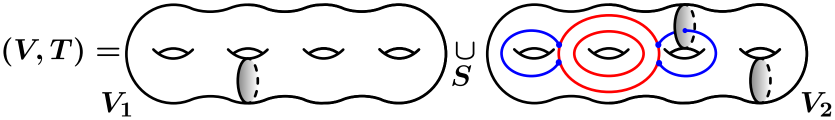


$$\mathrm{d}_S(T) := \min\left\{ \mathrm{d}_S(x,y) \text{ in } \mathcal{C}(S) \;\middle|\; \begin{array}{l} x : \partial \text{ of a c-disk in } V_1 \\ y : \partial \text{ of a c-disk in } V_2 \end{array} \right\}.$$



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Proposition.

$\forall g \geq 1$, $\forall n \geq 2$, $\forall t \geq 0$ and $\forall d > 0$, $\exists (V, T)$: a (g, n)-tangle with $\mathrm{tnl}(T) = t$ and $\mathrm{d}_S(T) > d$ for a Heegaard surface S of (V, T).

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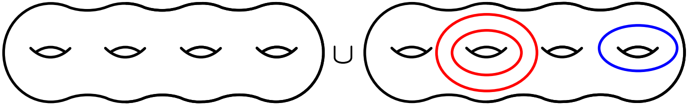

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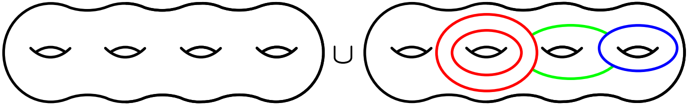

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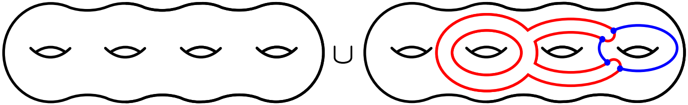

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The proof is based on the argument by Minsky-Moriah-Schleimer.

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Algebr. Geom. Topol. 7 (2007), 1471–1483.

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Key lemma. [Hempel (2001)]

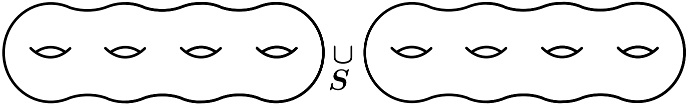
Suppose $X, Y \subset \mathcal{C}(S)$.

$\overline{X}, \overline{Y}$: closures of X, Y in $\mathcal{PML}(S)$.

Φ: a pseudo-Anosov map with stable/unstable laminations $\lambda^{\pm}$.

$$\lambda^{-} \notin \overline{Y} \text{ and } \lambda^{+} \notin \overline{X} \Longrightarrow \mathrm{d}(X, \Phi^{n}(Y)) \to \infty \text{ as } n \to \infty.$$

$S^3 = V \cup_S W$: a standard Heeg. splitting.



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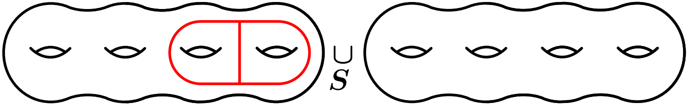
$V' := V - \mathbf{Nbd(Red\ Spine)}$.

Then $V' \cup_S W$ is a Heeg. splitting of a handlebody.



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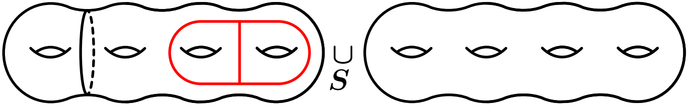
D: a disk cutting off a solid torus V'' from V'.



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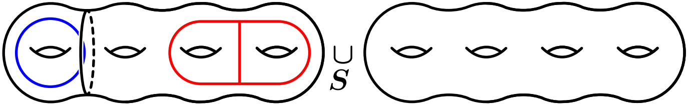

Existence of high distance knots

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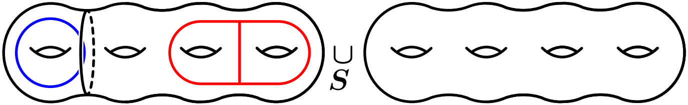
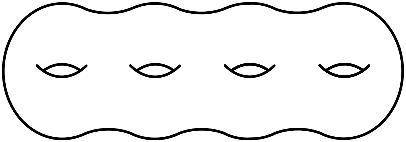


Need to find a "nice" pseudo-Anosov map on S extending over V'.

As Minsky-Moriah-Schleimer did, we take pants decompositions $\mathcal{P}$, $\mathcal{Q}$ as follows.



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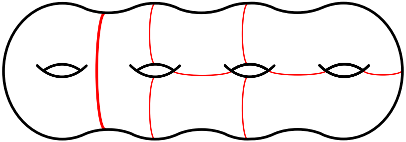

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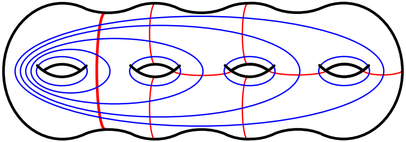

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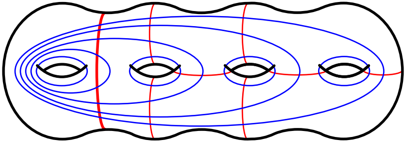
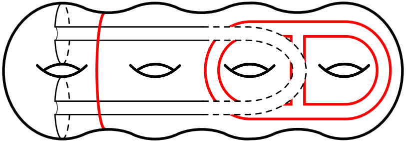


Claim 1. (Minsky-Moriah-Schleimer)
There is a meridian a of V' obtained from a band-sum on two copies of a meridian of V'' s.t. a traverses all seams of $\mathcal{P}$ and $\mathcal{Q}$ respectively.



Existence of high distance knots

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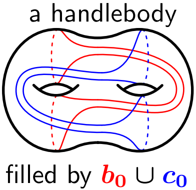
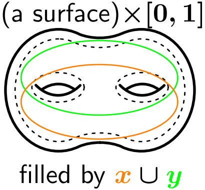
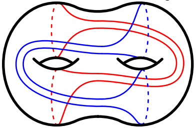




Existence of high distance knots

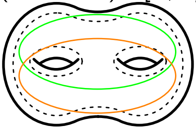
<u>Claim 2.</u> There are two meridians $\boldsymbol{b}, \boldsymbol{c}$ of $\boldsymbol{V'}$ s.t. $\boldsymbol{b} \cup \boldsymbol{c}$ fills $\boldsymbol{S}$.

a handlebody

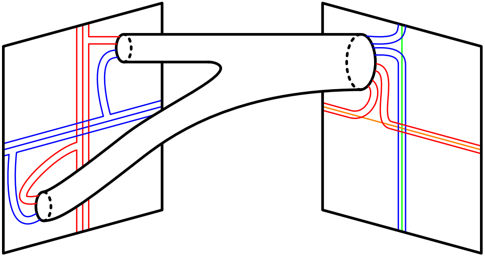


filled by $\boldsymbol{b_0} \cup \boldsymbol{c_0}$

(a surface)$\times[\mathbf{0}, \mathbf{1}]$



filled by $\boldsymbol{x} \cup \boldsymbol{y}$



$\boldsymbol{b}$: a "good" band sum of two copies of $\boldsymbol{b_0}$ along $\boldsymbol{x}$.

$\boldsymbol{c}$: a "good" band sum of two copies of $\boldsymbol{c_0}$ along $\boldsymbol{y}$.

Claim 3. (Minsky-Moriah-Schleimer)
$\Phi_N := \tau_a^N \circ (\tau_b \circ \tau_c^{-1}) \circ \tau_a^{-N}$ $(N \gg 0)$ satisfies the assumption of Key lemma.

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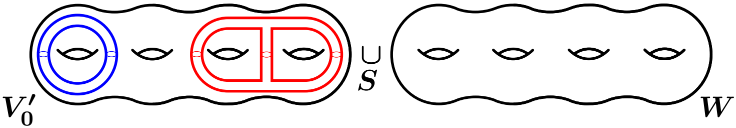
Hence $\boxed{\mathrm{d}(\Phi_N^n(\mathcal{D}(V_0')), \mathcal{D}(W)) \to \infty \text{ as } n \to \infty.}$



Outline of proof of Theorem 2

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Theorem 2.

$\forall g \in \mathbb{Z}_{\geq 1},\ \forall n \in \mathbb{Z}_{\geq 2},\ \forall t_1, \forall t_2 \in \mathbb{Z}_{\geq 0},\ \exists K = T_1 \cup T_2$: a (g, n)-tangle decomposition s.t. $\mathrm{tnl}(T_1) = t_1$, $\mathrm{tnl}(T_2) = t_2$ and

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Remark.

If a generalized tangle contains an *essential* surface, its Euler characteristic bounds distance from above.

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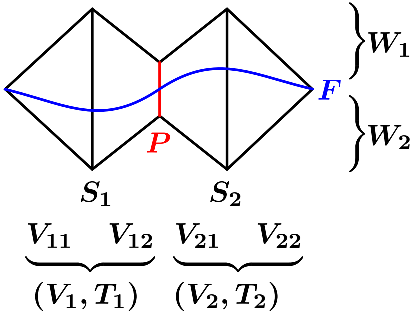
$(M^3, K) = (V_1, T_1) \cup_P (V_2, T_2)$
F : a minimal genus Heegaard surface of (M^3, K).



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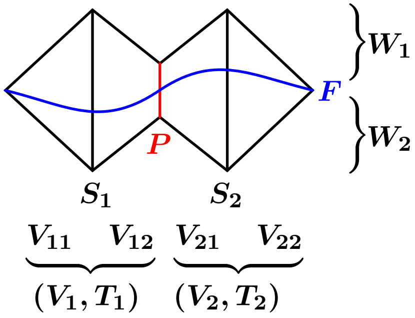

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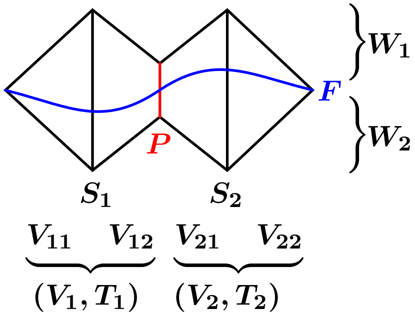

Proposition.

F is weakly reducible.

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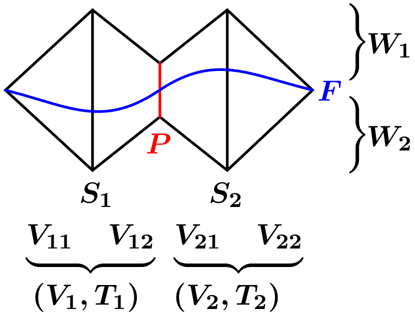

Proposition.

F is weakly reducible.

(proof)

Kobayashi-Qiu's argument : otherwise, $F \cap V_1$ or $F \cap V_2$ is essential.

Essential surfaces bound distance from above, a contradiction.

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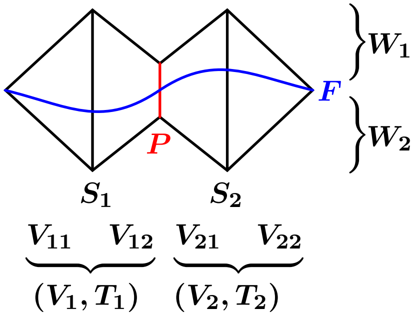

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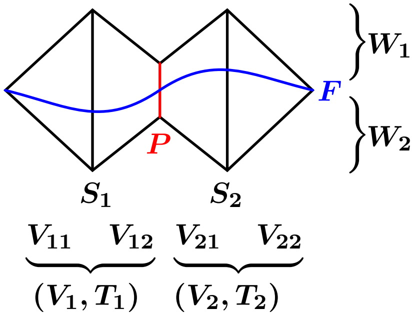

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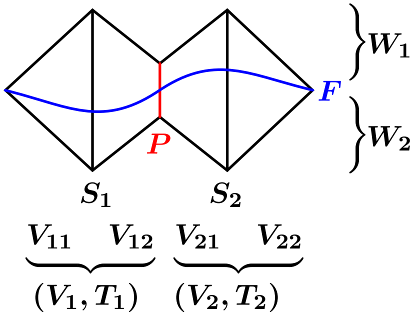

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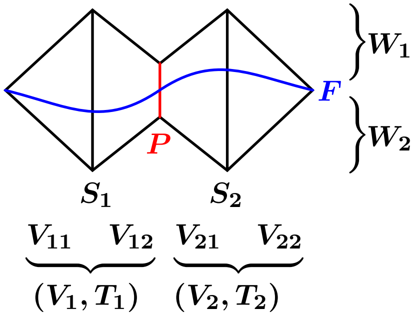

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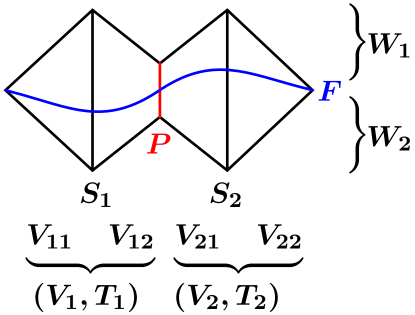

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