On zeros of the Alexander polynomials of alternating knots

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Conjecture (Hoste, 2002)

K: an alternating knot

$$\Delta_K(t) = 0 \quad \Rightarrow \quad \mathrm{Re}\, t > -1.$$

Diagram

Topology

"alternating" ⟷ (roots of) Alex. polyn.

relation?

Hoste conj.

Conjecture (Hoste, 2002)

K: an alternating knot

$$\Delta_K(t) = 0 \quad \Rightarrow \quad \mathrm{Re}\, t > -1.$$

$t \in \mathbb{C}$: a root of $\Delta_K(T)$

- alternating property (Murasugi ('58), Crowell ('59)) $\rightsquigarrow \quad t \in \mathbb{R} \Rightarrow t > 0.$

For the 2-bridge knots,

- Lyubich-Murasugi ('12): $-3 < \mathrm{Re}\, t < 6.$
- Stoimenow ('18), Koseleff-Pecker ('15):
  $$|t^{1/2} - t^{-1/2}| < 2. \rightsquigarrow \mathrm{Re}\, t > -3/2.$$
- I. (('17)): $\mathrm{Re}\, t > -1.$

Results

Conjecture (Hoste, 2002)

K: an alternating knot

$$\Delta_K(t) = 0 \quad \Rightarrow \quad \mathrm{Re}\, t > -1.$$

Thm A Hoste's conjecture holds for the alternating-pretzel knots (links).

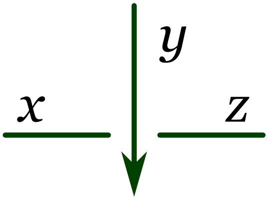
Thm B Hoste's conjecture does NOT hold for infinitely many alternating Montesinos knots.

Thm C L: an alternating Montesinos link

$$\Delta_L(t) = 0 \quad \Rightarrow \quad \mathrm{Re}\, t > -2.$$

Contents

· a $\mathbb{C}_t$-**coloring** is $\mathcal{C} : \{\text{arcs}\} \to \mathbb{C}$ s.t.



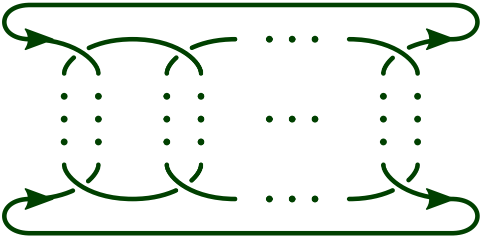
$$\begin{aligned} \mathcal{C}(z) &= t\mathcal{C}(x) + (1-t)\mathcal{C}(y). \\ (\Leftrightarrow \mathcal{C}(x) &= t^{-1}\mathcal{C}(z) + (1-t^{-1})\mathcal{C}(y).) \end{aligned}$$

Rem $\mathbb{C}_t$ is a quandle.

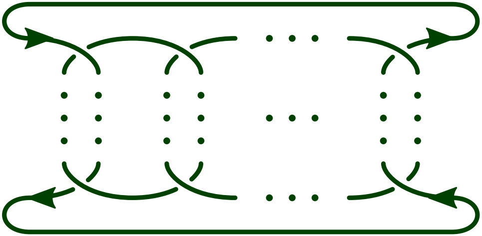
- (Inoue, '01) $^{\exists}$non-triv. $\mathbb{C}_t$-col. $\Leftrightarrow \Delta_K(t) = 0$.

alternating-pretzel links:

type I:



type II:



alternating-pretzel links:

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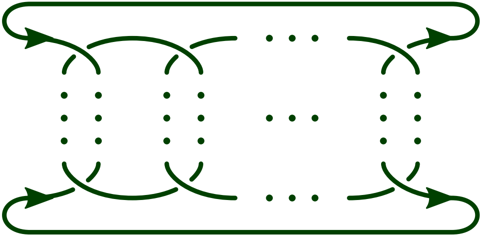
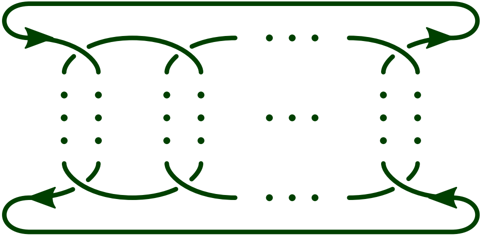
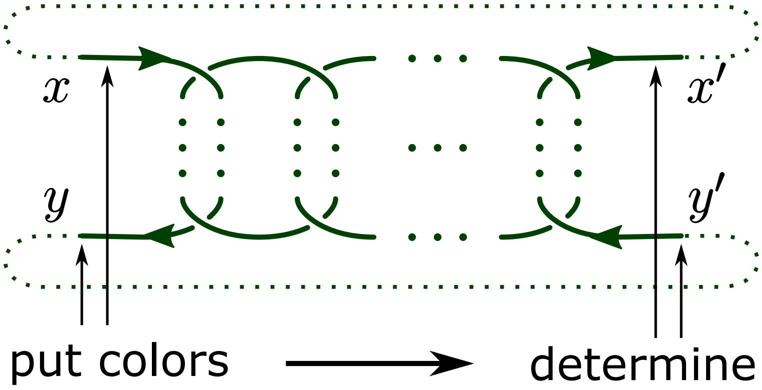


type II:

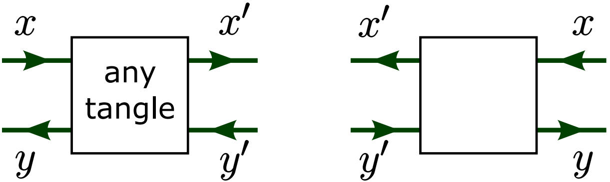


Strategy of locating roots

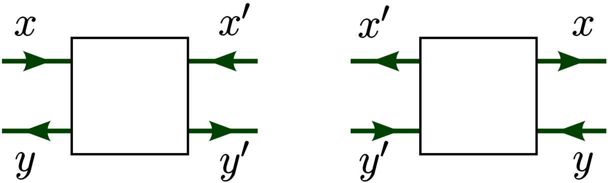


$$\begin{cases} x = x' \\ y = y' \end{cases} \quad \Leftrightarrow \quad \Delta_L(t) = 0.$$

Properties of colored tangles



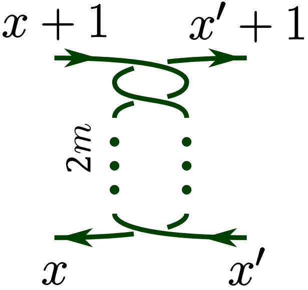
$$\rightsquigarrow \quad x - y \quad = \quad x' - y'$$



$$\rightsquigarrow \quad (-t)(x - y) \quad = \quad x' - y'$$

Parametrization

$\sigma_1(m)$:

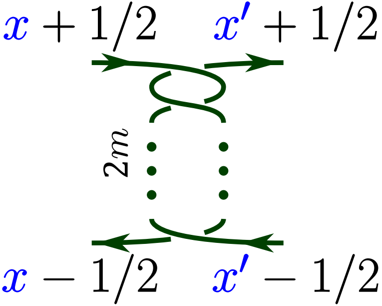


- $x' = x + c_{\sigma_1(m)}$,

$$c_{\sigma_1(m)} = \frac{1}{m(1+s^{-1})}, \qquad \text{where } s = -t.$$

Parametrization

$\sigma_1(m)$:



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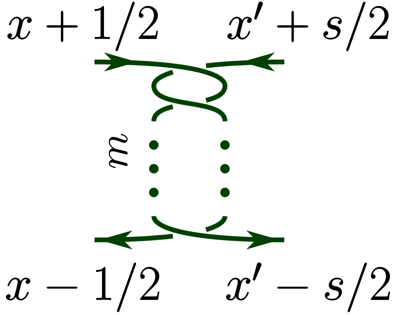
Parametrization

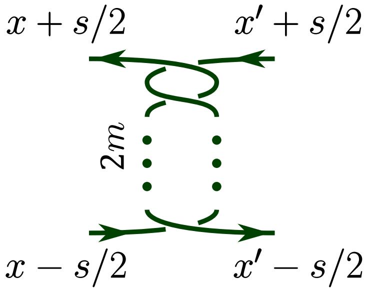
$\sigma_2(m)$:



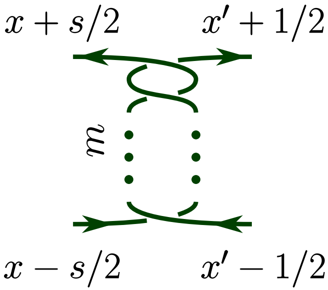
- $x' = x + c_{\sigma_2(m)}$,

$$c_{\sigma_2(m)} = \frac{s-1}{2} \cdot \frac{s^m + 1}{s^m - 1}, \qquad \text{where } s = -t.$$

Parametrization



$-\sigma_1(m)$



$-\sigma_2(m)$

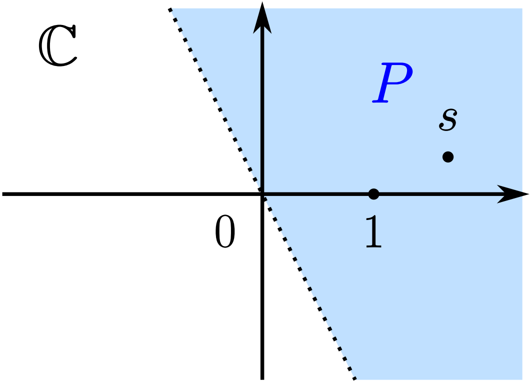
- $x' = x + c_{-\sigma_i(m)}$ for $i = 1, 2.$

<u>Lem</u> $c_{-\sigma_i(m)} = c_{\sigma_i(m)}$ for $i = 1, 2.$

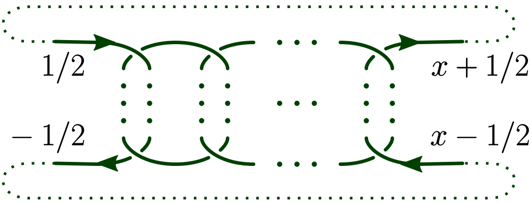
c_σ are "one-sided"

$$P := \{z \in \mathbb{C} \mid \operatorname{Re} x/(s-1) > 0\}.$$

Lem $\quad \operatorname{Re} t \leq -1 \quad \Rightarrow \quad c_{\sigma_1(m)}, c_{\sigma_2(m)} \in P.$



Proof of Theorem A



$$t \in \mathbb{C}, \quad \operatorname{Re} t \le -1.$$

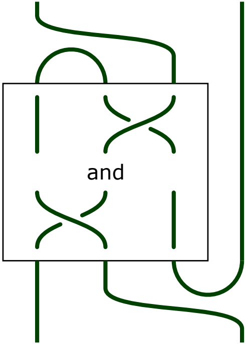
$$x = c_{\pm\sigma_{i_1}(m_1)} + \cdots + c_{\pm\sigma_{i_k}(m_k)} \overset{\text{Lem}}{\in} P.$$

$\rightsquigarrow x \neq 0$, i.e. $\Delta_L(t) \neq 0$.

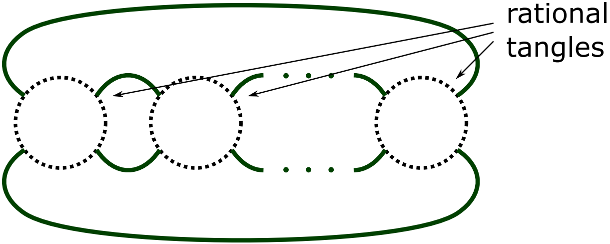
Theorem A

Hoste's conjecture holds for the alternating-pretzel links.

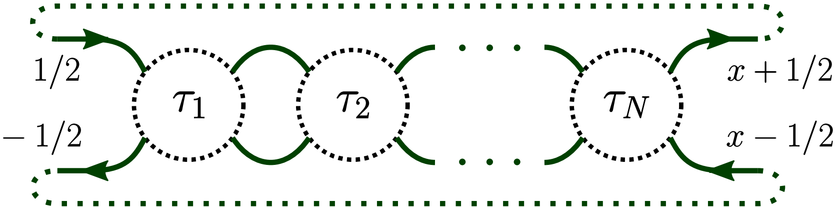
(a diagram of)
a **rational tangle**:



a **Montesinos knot/link**:



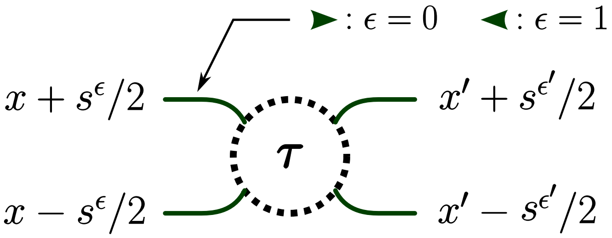
Strategy



put colors ⟶ x is determined

$$x = 0 \quad \Leftrightarrow \quad \Delta_L(t) = 0.$$

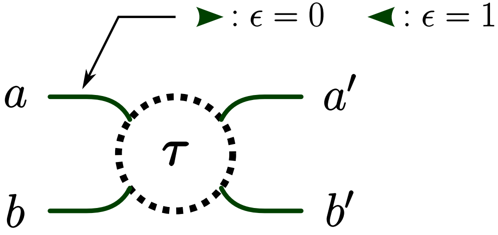
Parametrization



- $\exists c_\tau$: a rational func. on t s.t. $x' = x + c_\tau$.

Parametrization

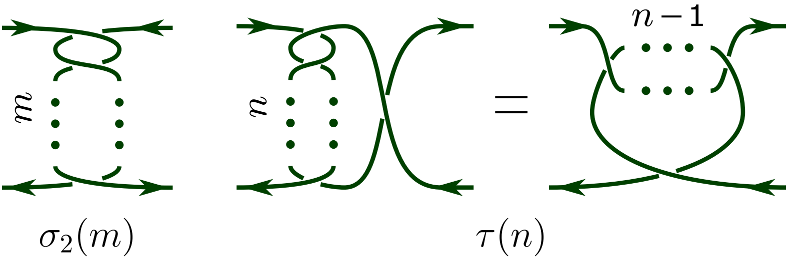
∀nontriv. col.:



$$\Rightarrow \quad c_\tau = s^\epsilon \cdot \frac{\frac{a'+b'}{2} - \frac{a+b}{2}}{a-b}.$$

Lem $c_\tau = c_{-\tau}$.

Ex.

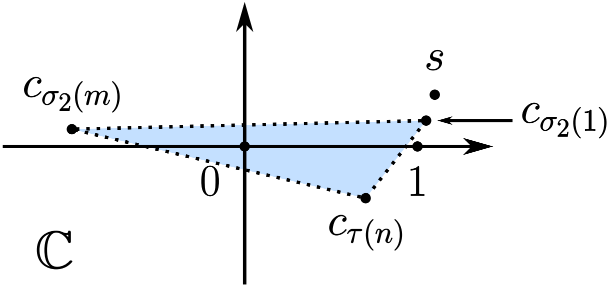


$$c_{\sigma_2(m)} = \frac{1-s}{2} \cdot \frac{1+s^m}{1-s^m}, \qquad c_{\tau(n)} = \frac{s-s^n}{1-s^n}.$$

Are c_τ “one-sided”?

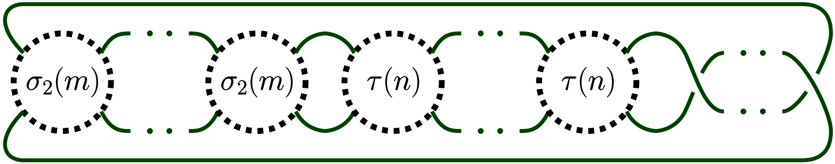
Are c_τ "one-sided"? —NO! (when $\operatorname{Re} t \leq -1$)

<u>Lem</u> $\exists m, n, t \ (\operatorname{Re} t < -1)$ s.t. the triangle spanned by $\underbrace{c_{\sigma_2(1)}(t), c_{\sigma_2(m)}(t), c_{\tau(n)}(t)}_{=: \Delta(m,n,t)}$ contains $0 \in \mathbb{C}$:



<u>Ex.</u> $(m, n, t) = (51, 64, -1.005 - 0.099i)$

Candidate links:



Theorem B

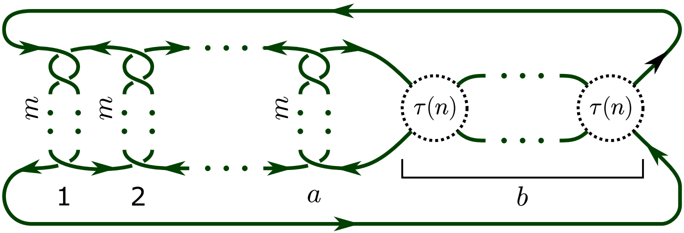
$$\left\{ t \in \mathbb{C} \;\middle|\; \begin{array}{l} \mathrm{Re}\, t < -1, \\ \exists \text{alt.Mont. link } L \\ \text{s.t. } \Delta_L(t) = 0 \end{array} \right\} \supset \left(\bigcup_{d \in \mathbb{Z}_{>0}} \mathbb{Q}(\sqrt{-d}) \right) \cap U,$$

where U is the nonempty open set

$$\bigcup_{m,n \in \mathbb{Z}_{>0}} \{ t \in \mathbb{C} \mid \mathrm{Re}\, t < -1,\ 0 \in \Delta(m,n,t) \}.$$

Counterexamples (j.w. M. Hirasawa and M. Suzuki)

Candidate link $L(m, a, n, b)$:



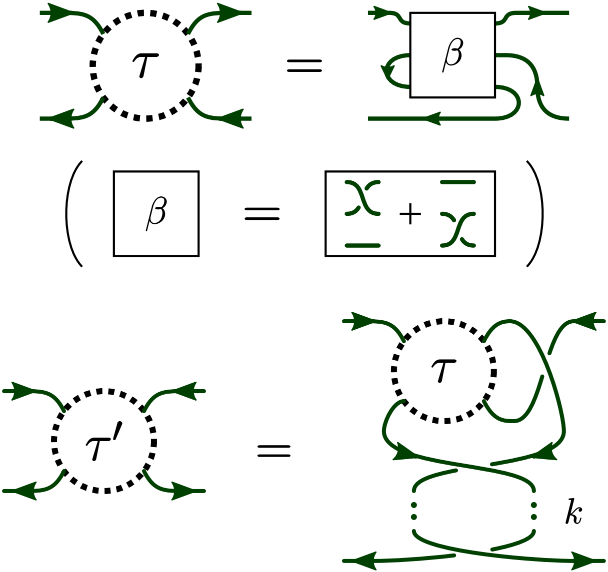
m : odd, a, n : even, $b = 1 \quad \Rightarrow \quad L(m, a, n, b)$: knot

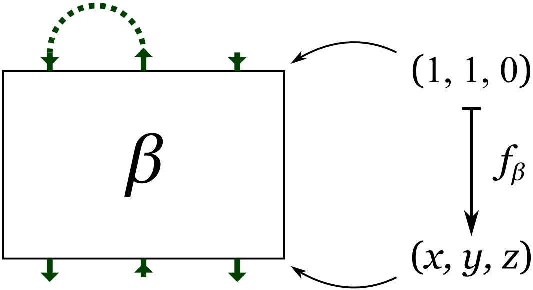
45 counterexamples among 410141 candidates up to 1000 crossings:

$(m, a, n) =$(39, 24, 56), (41, 20, 56), (41, 22, 58), (43, 18, 58), (43, 20, 58), (43, 20, 60), (45, 18, 60), (45, 20, 60),(45, 20, 62), (47, 16, 60), (47, 16, 62), (47, 18, 62), (47, 18, 64), (49, 16, 62), (49, 16, 64), (49, 18, 64),(49, 18, 66), (51, 14, 64), (51, 16, 64), (51, 16, 66), (51, 18, 66), (51, 18, 68), (53, 14, 66), (53, 16, 66),(53, 16, 68), (55, 14, 68), (55, 16, 68), (55, 16, 70), (57, 14, 70), (57, 16, 70), (57, 16, 72), (57, 16, 74),(59, 14, 72), (59, 14, 74), (61, 14, 74), (61, 14, 76), (63, 14, 76), (63, 14, 78), (65, 14, 78), (65, 14, 80),(71, 12, 84), (73, 12, 84), (73, 12, 86), (75, 12, 86), (75, 12, 88).

min. crossing: $L(51, 14, 64, 1)$ (778 crossings, $\mathrm{Re}\, t \sim -1.000030$)

A rational tangle is presented by an oriented 3-braid:



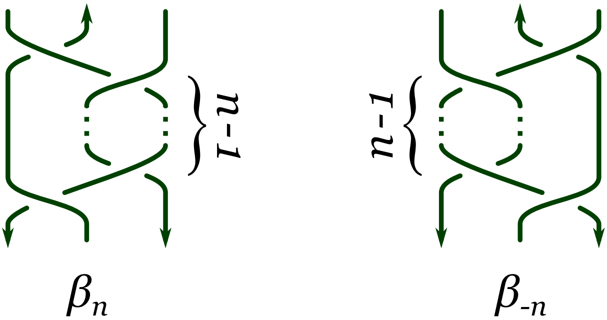


$$c_\tau = \xi = -z/x,$$

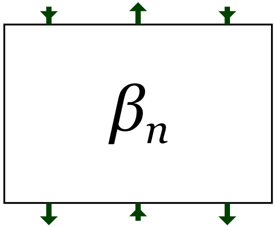
$$c_{\tau'} = g_k(\xi) = \frac{s-1}{2} \cdot \frac{(1+s^{-k})\xi + (s+s^{-k})}{(1-s^{-k})\xi + (s-s^{-k})}.$$

Lem β is decomposed into β_n:

$$\beta = \beta_{n_1} \cdots \beta_{n_k}.$$



$(n = 1, 2, \cdots)$

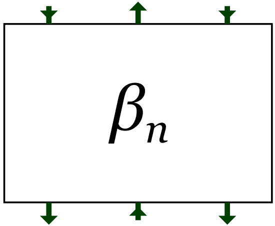


$$(x, y, z) \mapsto (x', y', z')$$

$$\left(\text{e.g.} \quad x' = \frac{t(x-y+z) + x + t^{-1}(y-x) + (-t)^n(z-x)}{1+t}. \right)$$

- $x - y + z = x' - y' + z'$.

  $\rightsquigarrow x - y + z = 0 \quad \Rightarrow \quad x' - y' + z' = 0.$



$$(x, y, z) \mapsto (x', y', z')$$

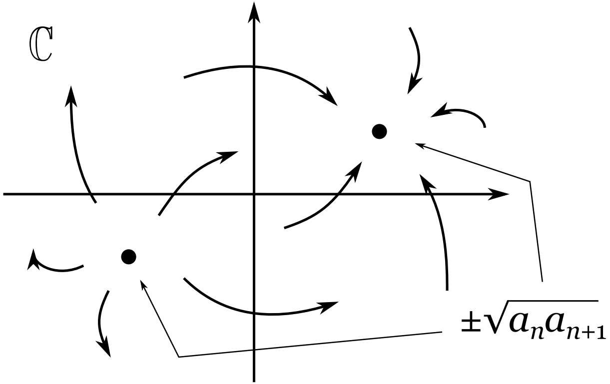
$$\xi := \text{-}z/x \quad \overset{f_n}{\dashrightarrow} \quad \xi' := \text{-}z'/x'$$

Lem $\xi' = \dfrac{\xi + a_n}{\xi/a_{n+1} + 1} \; (=: f_n(\xi)),$

where $\quad a_n := \dfrac{s^n - s}{s^n - 1}, \quad s = -t.$

What are f_n like?

$$f_n(\xi) = \frac{\xi - a_n}{-\xi/a_{n+1} + 1} \qquad a_n = \frac{s^n - s}{s^n - 1} \qquad (\operatorname{Re} s \geq 1)$$



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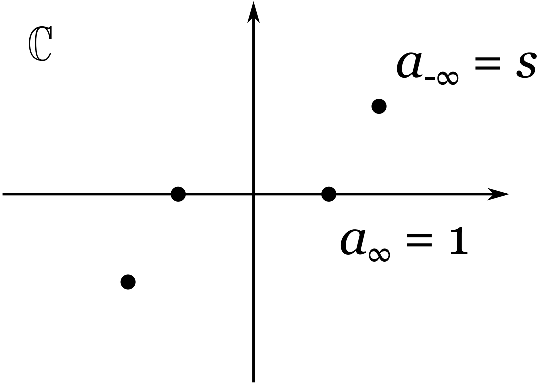

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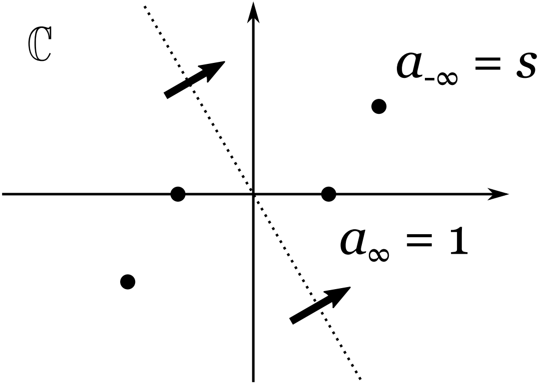
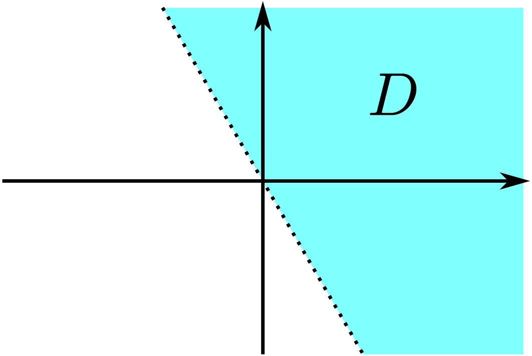


Assume $\operatorname{Re} s \geq 1,\ (\operatorname{Im} s \geq 0,)\ s \neq 1.$

<u>Lem</u> $^{\exists}D \subset \mathbb{C}$: an open half plane s.t.

$$\begin{cases} f_n(\overline{D}) \subset D \ (n \in \mathbb{Z}\backslash\{0,1\}), \\ f_1(D) \subset D. \end{cases}$$



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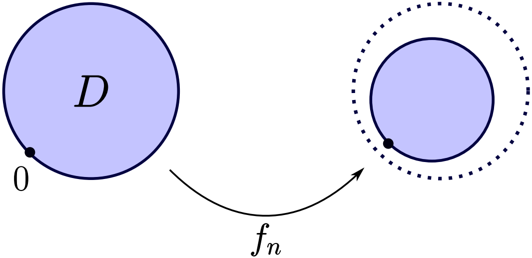
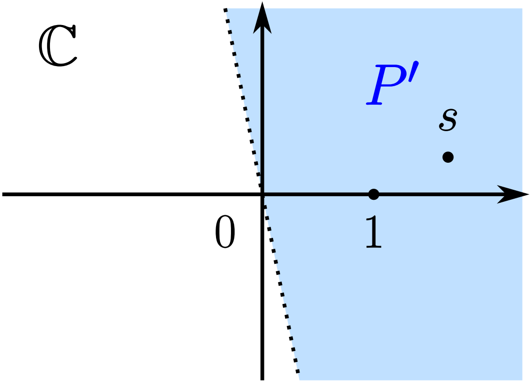


$\rightsquigarrow c_\tau = f_{n_k} \circ \cdots \circ f_{n_1}(0) \in D : c_\tau$ are "one-sided".

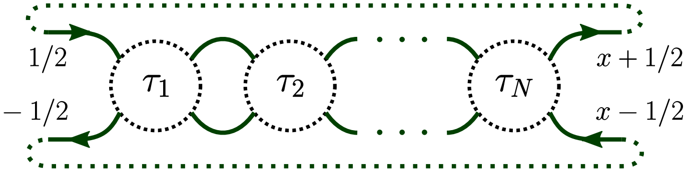
$c_\mathcal{T}, c_{\mathcal{T}'}$ are "one-sided" (when $\operatorname{Re} t \leq -2$)

$$P' := \{z \in \mathbb{C} \mid \operatorname{Re} x/s > 0\}.$$

<u>Lem</u> $\operatorname{Re} t \leq -2 \quad \Rightarrow \quad c_\mathcal{T}, c_{\mathcal{T}'} \; (= g_k(c_\mathcal{T})) \in P'.$



Proof of Theorem C



$$t \in \mathbb{C}, \quad \operatorname{Re} t \leq -2.$$

$$x = c_{\tau_1} + \cdots + c_{\tau_N} \overset{\text{Lem}}{\in} P'.$$

$\rightsquigarrow x \neq 0$, i.e. $\Delta_L(t) \neq 0$.

Theorem C

L: an alternating Montesinos link

$$\Delta_L(t) = 0 \quad \Rightarrow \quad \operatorname{Re} t > -2.$$

Future problems

- Is there a (lower) bound of (the real parts of) the roots of the Alexander polynomials for the alternating knots? If there is, find it.
- Find the infimum of the real parts of the roots for the alternating Montesinos knots.
- Characterize (the closure of the set of) the roots for the 2-bridge knots.
- Can we say something about the roots of other polynomial invariants (e.g. Jones polyn.)?