From one-cone tori to two-bridge cone manifolds

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Topology and Geometry of Low-dimensional Manifolds
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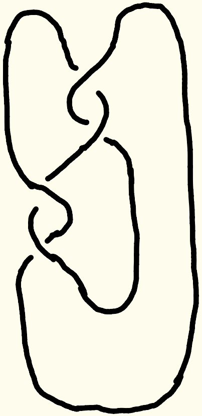
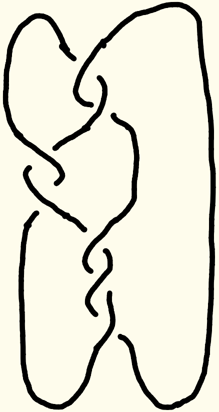
Outline

Part 1. From once-punctured torus
to two-bridge knot complements
(With Sakuma, Wada, Yamashita)

Part 2. From one-cone torus
to two-bridge cone manifolds
(Partly joint with Yamashita)

2-bridge knots (and links)





$$\frac{1}{2+\frac{1}{2}} = \frac{2}{5} \qquad \frac{1}{2+\frac{1}{2+\frac{1}{3}}} = \frac{7}{17}$$

$S(5,2) = C(2,2)$ $\quad$ $S(17,7) = C(2,2,3)$

$\frac{q}{p} \in \mathbb{Q}$

$S(p,q) \sim S(p',q')$

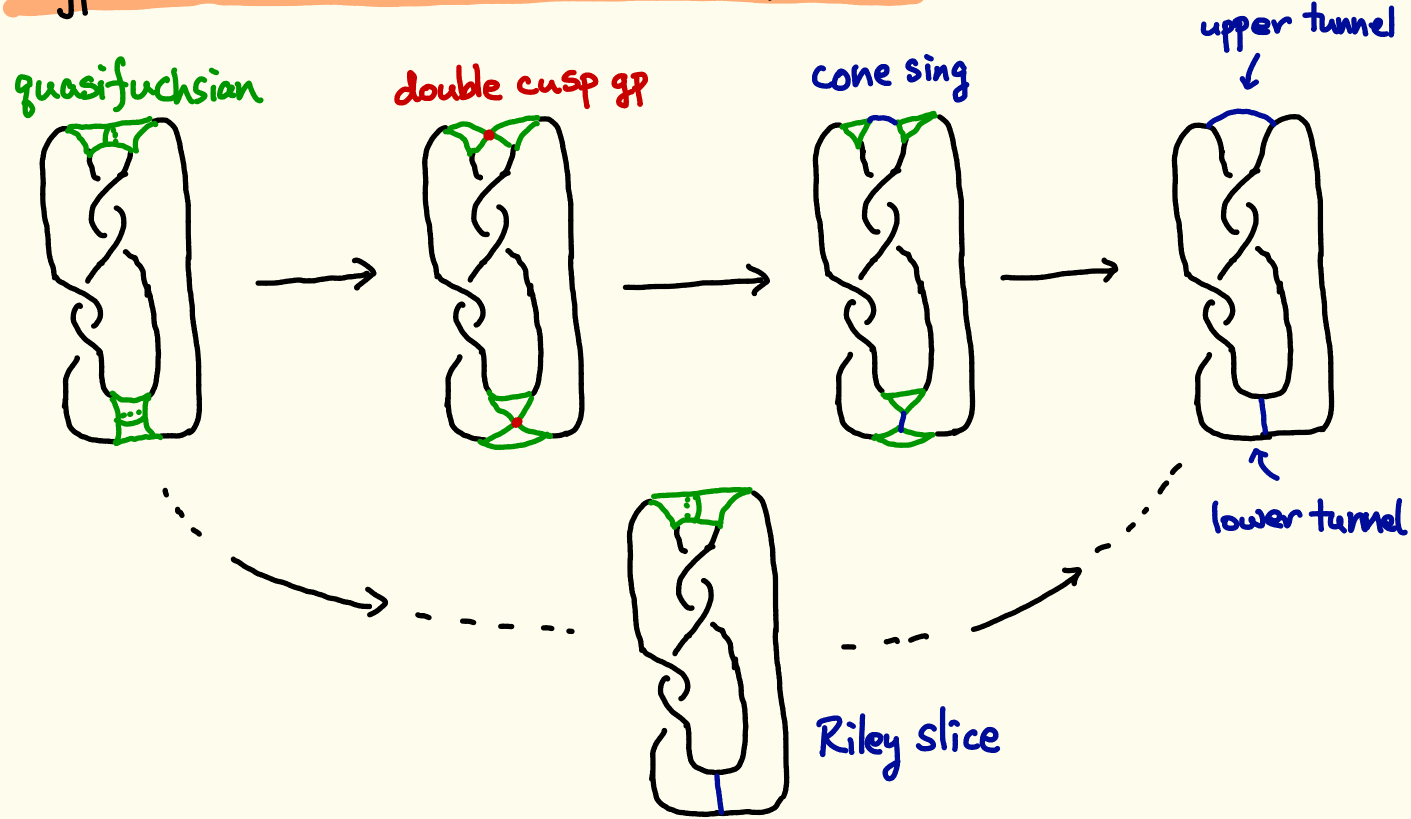
(up to mirror image)

with $1 \leq q' \leq p/2$

* $S(p,q)$: hyperbolic

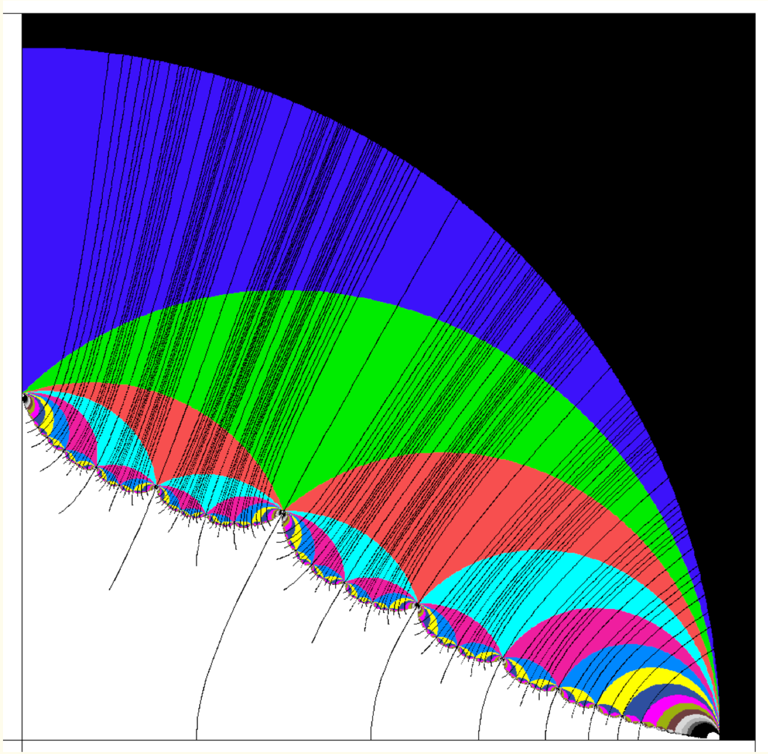
$\Leftrightarrow \quad q' \neq 1$

Hyperbolic str of 2-br knot complement — ASWY picture



✩ The Ford domains are characterized !!

Riley slice



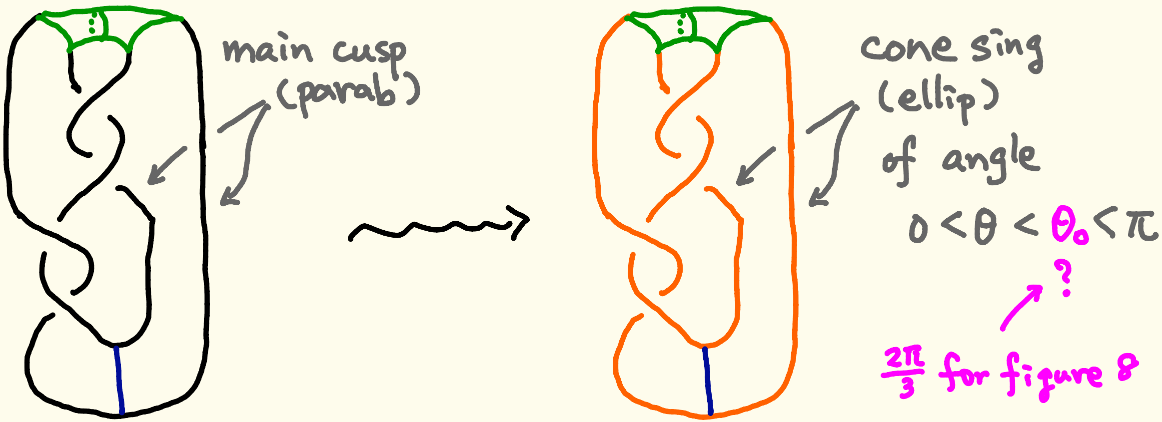
Yamashita's output of Riley slice

☆ There is a practical "algorithm" to check if a given two-parabolic group is discrete or not!!

Demonstration 1

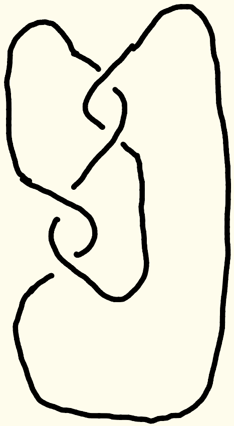
The ongoing project

Goal: Replace the "main cusp" with cone singularity.

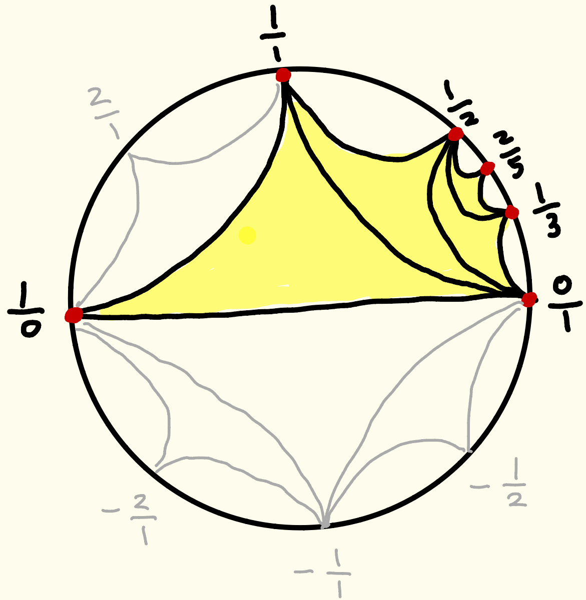


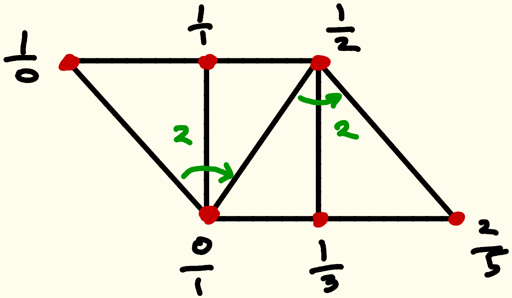
Rem The double covering space of S^3 branched along $S(p,q)$ is the lens space $L(p,q)$.

Farey tessellation

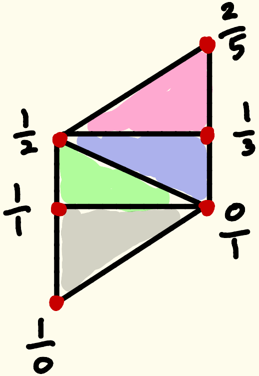


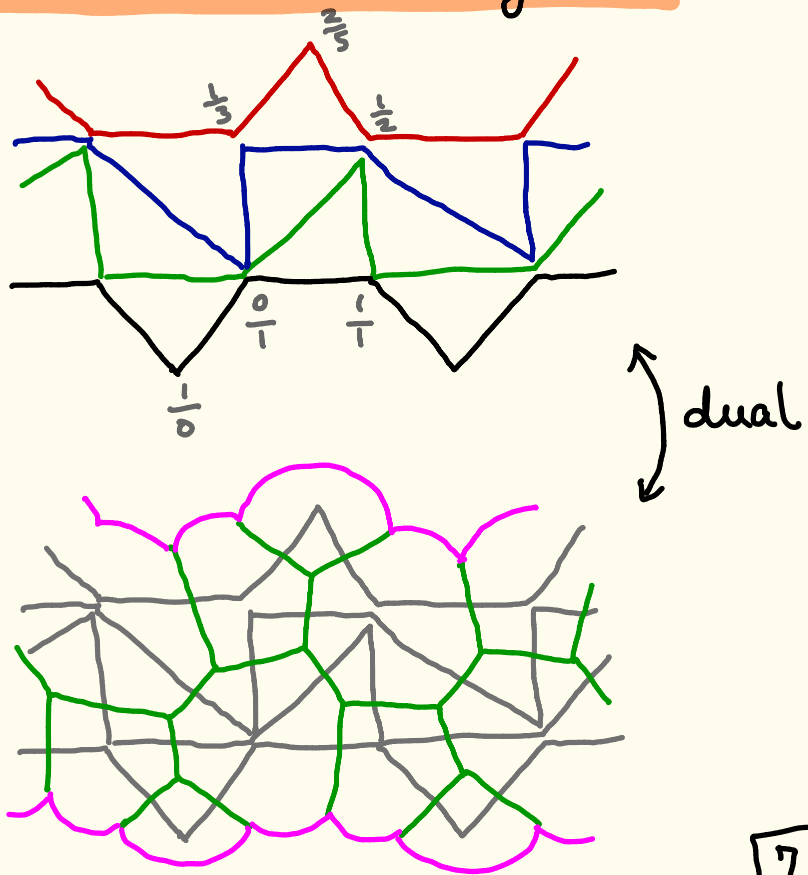
$S(5,2) = C(2,2)$





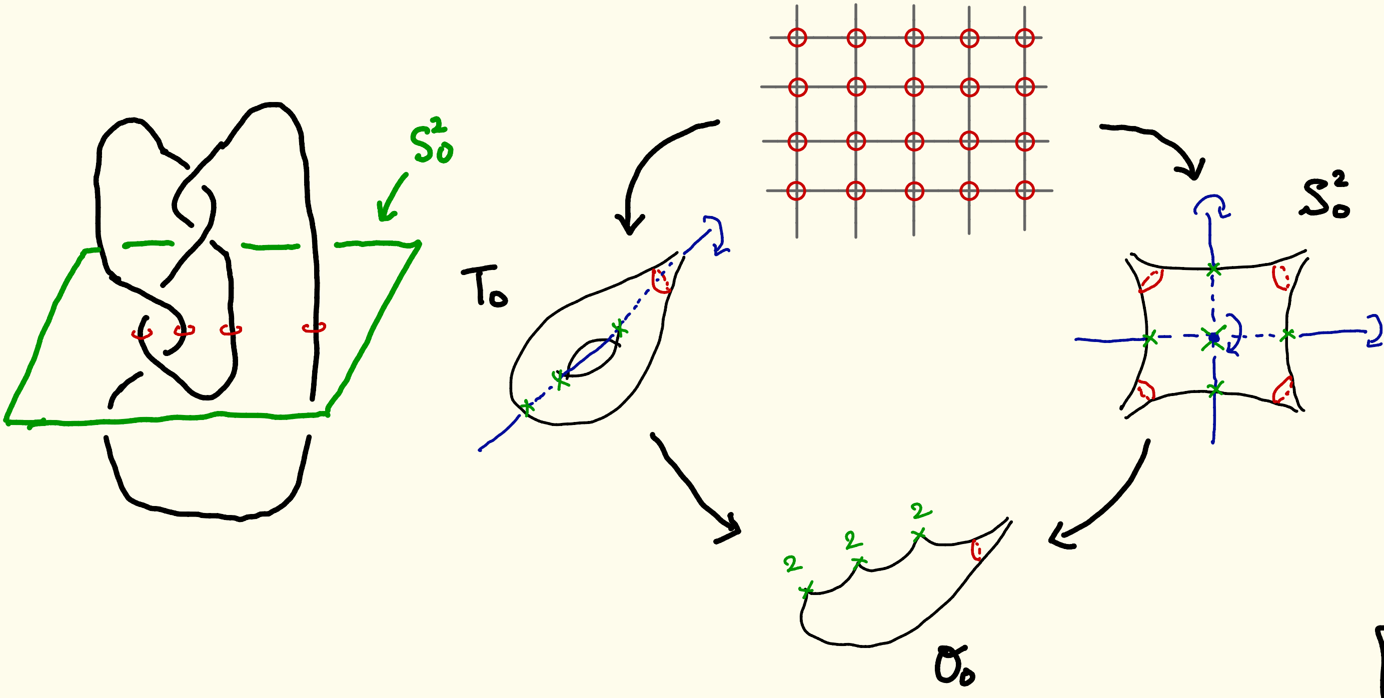
Description of comb str due to Jorgensen





Idea of proof — based on Jorgensen's argument

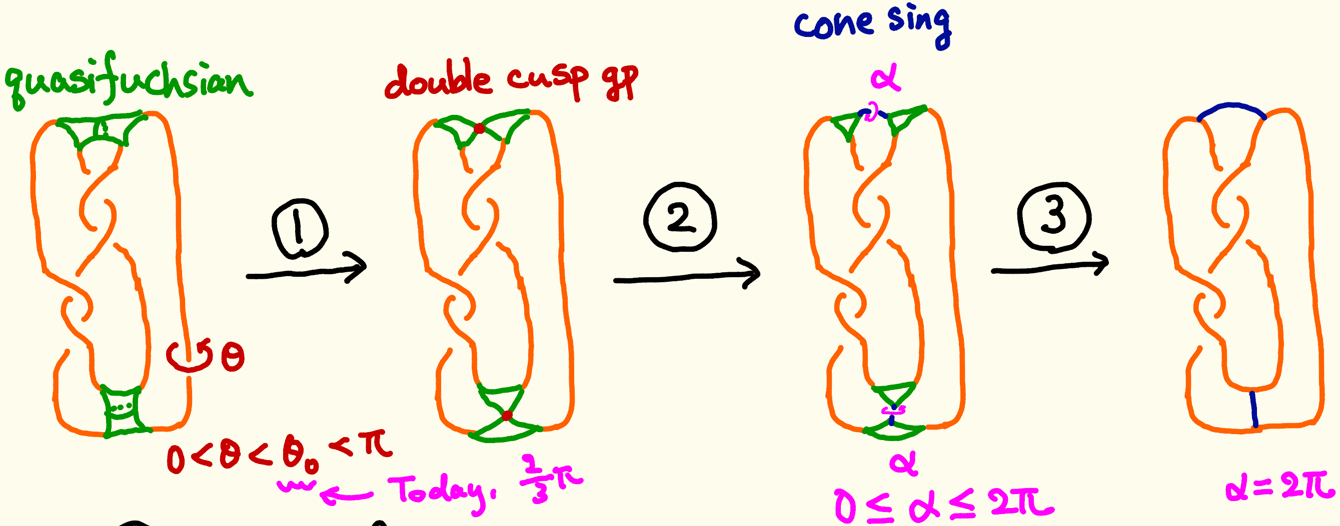
☆ Jorgensen characterized the combinatorial structures of the Ford domains for punctured torus groups.



Part 2

From one-cone torus to two-bridge cone manifolds

Steps on the way



① Quasifuchsian

②, ③ Additional cone singularity

---- Something happens !!

Demonstration 2

Quasifuchsian structure

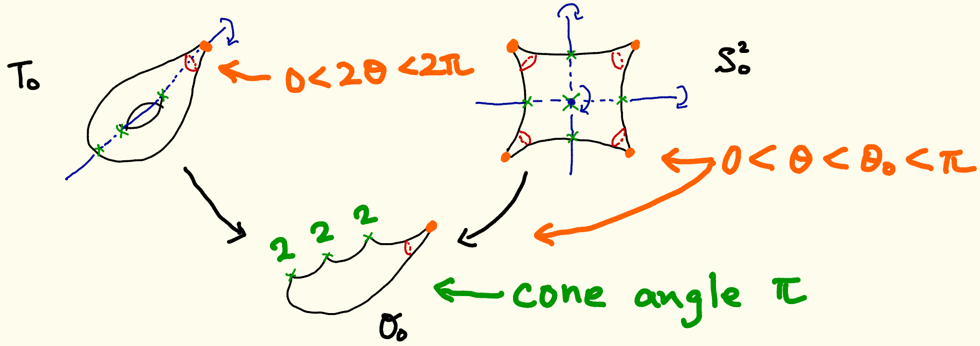
[Moroianu-Schlenker, Lecuire-Schlenker]

"Quasifuchsian manifold with particle"

$$(M, \Sigma) = (\text{surface}, \{\text{fin. pts}\}) \times \mathbb{R}$$

- all cone angles $< \pi$
- admits compact convex core

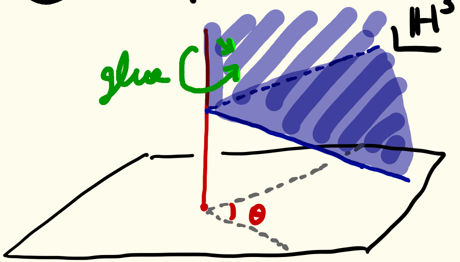
Thm [L-S] $QF(M) \cong \mathrm{Teich}(\partial M)$



Ford domain for cone hyp str

Suppose $\exists C \subset M$

s.t. $C \cong$

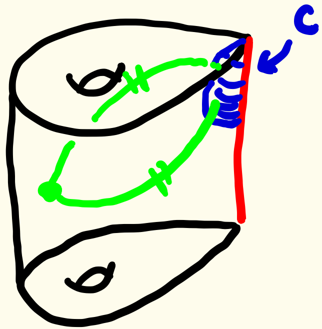


(horoball with cone)

The Ford domain w.r.t. C is

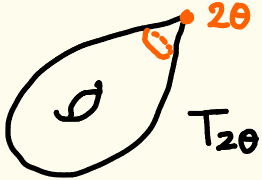
$$\overline{M - \left\{ x \,\middle|\, \begin{array}{l} \exists 2 \text{ shortest paths} \\ \text{from } x \text{ to } C \end{array} \right\}}$$

cut locus w.r.t. C



Partial results

Suppose $0<\theta<\pi$ ($\Rightarrow 0<2\theta<2\pi$)



Thm 1 [A. 2015]

The Ford domain for any "fuchsian" structure on $T_{2\theta}\times\mathbb{R}$ has "good" comb. str.

Thm 2 [A. 2018]

- The Ford domain for any "thin" structure on $T_{2\theta}\times\mathbb{R}$ has "good" comb. str.
- The space of thin str is parametrized by the hyp str of ∂(compact convex core).

Explosion of isometric hemispheres

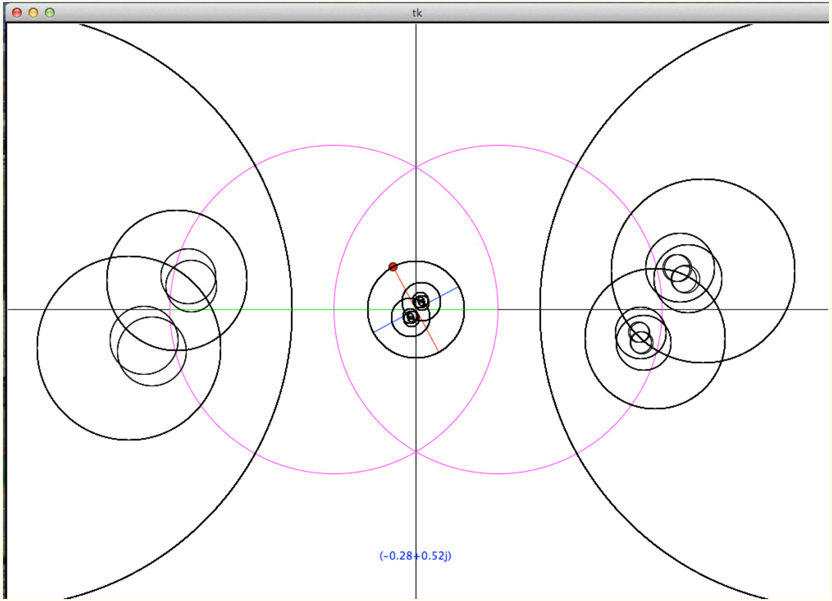
In Demo 2, $r(\text{ison hemi}) \longrightarrow \infty$ as $\alpha \longrightarrow 2\pi - \theta$.

- $\theta = 0$ (cusp case)
  Explosion occurs at $\alpha = 2\pi$, where we obtain the complete hyp str of 2-br knot.

- $\theta > 0$ (our new case)
  Explosion occurs before $\alpha = 2\pi$
  $\Rightarrow$ We cannot reach the desired str only via Ford domains

Employing Dirichlet domain

Series - Tan - Yamashita studied the extended Riley slices, where Dirichlet domain is employed.



"Surely, the comb str are same as those of Ford dom for the Riley slice!!"

— said Yamashita

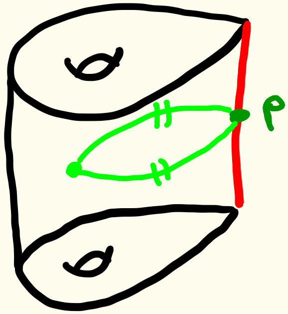
Demonstration 3

Dirichlet domain

The Dirichlet domain w.r.t. p is

$$\overline{M - \left\{ x \,\middle|\, \begin{array}{c} \exists_2 \text{ shortest paths} \\ \text{from } x \text{ to } p \end{array} \right\}}$$

cut locus w.r.t. p



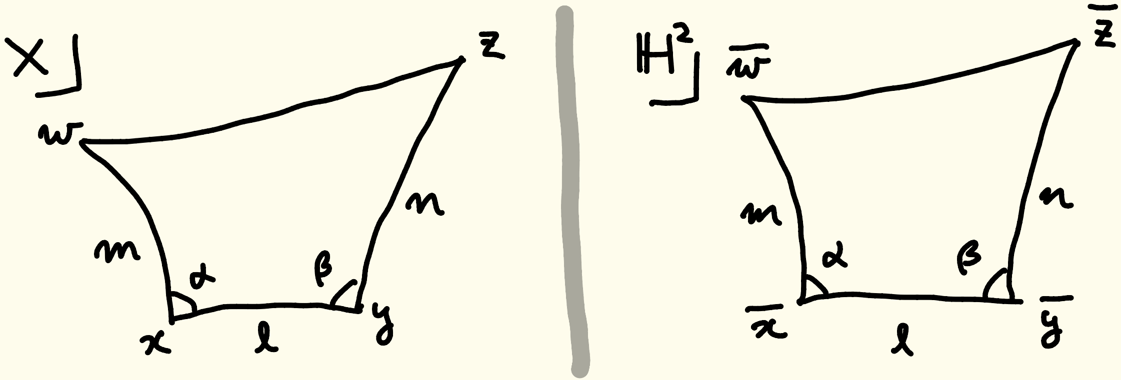
Prop $(M, \Sigma) = (S.\{\text{finite pts}\}) \times \mathbb{R}$, cone angle $\leq \pi$

Suppose $\left\{ \begin{array}{l} \cdot\ M \text{ contains horoballs with cone at} \\ \quad \text{both ends of } \Sigma \\ \cdot\ M \text{ contains a compact convex core} \end{array} \right.$

$$\Longrightarrow \# \left\{ \begin{array}{c} \text{comb str of } D_p \\ \text{w.r.t. } p \in \Sigma \end{array} \right\} < \infty$$

Key Lemma for Prop

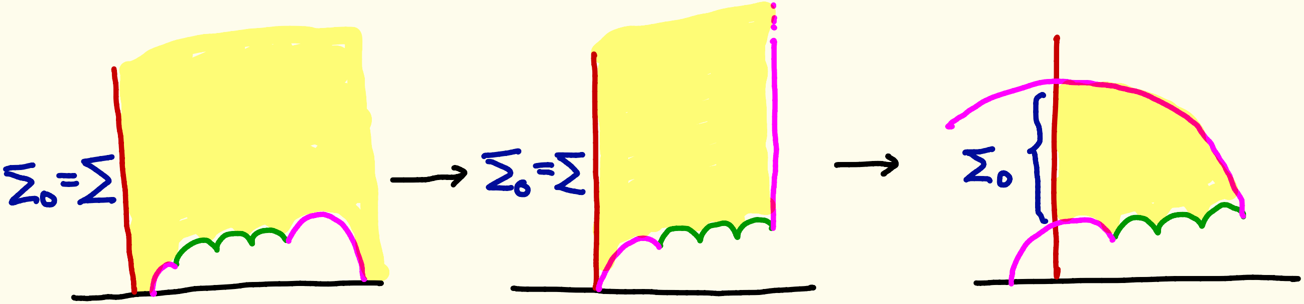
Lemma Let X be a CAT(-1) space.
Then we have the following comparison:



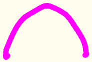
$$\alpha, \beta \geq \frac{\pi}{2} \implies d_X(w, z) \geq d_{\mathbb{H}^2}(\bar{w}, \bar{z})$$

Choice of base points

Schematic sectional views of Ford/Dirichlet domains:



$\alpha < 2\pi - \theta$ $\quad$ $\alpha = 2\pi - \theta$ $\quad$ $2\pi - \theta < \alpha \leq 2\pi$

 symbolize the axes of "elliptic generators" of slope $\frac{1}{0}$, $\frac{q}{p}$ for $K(p, q)$.

Conj $\forall p \in \Sigma_0$, the Dirichlet domain w.r.t. p has good comb str.

Demonstration 4

Some comments

- The cone hyp str for the figure 8 knot in Demo 4 are studied by Hilden-Lozano-Montesinos in detail by using Dirichlet domains w.r.t. other base points.
- Another family — for bundle over S^1 — is studied by Jorgensen, and Heusener-Porti-Suárez.
- The families for figure 8 knot are also studied by Thurston.