Akira Yasuhara (Tsuda University)

Title. Arrow calculus for welded links

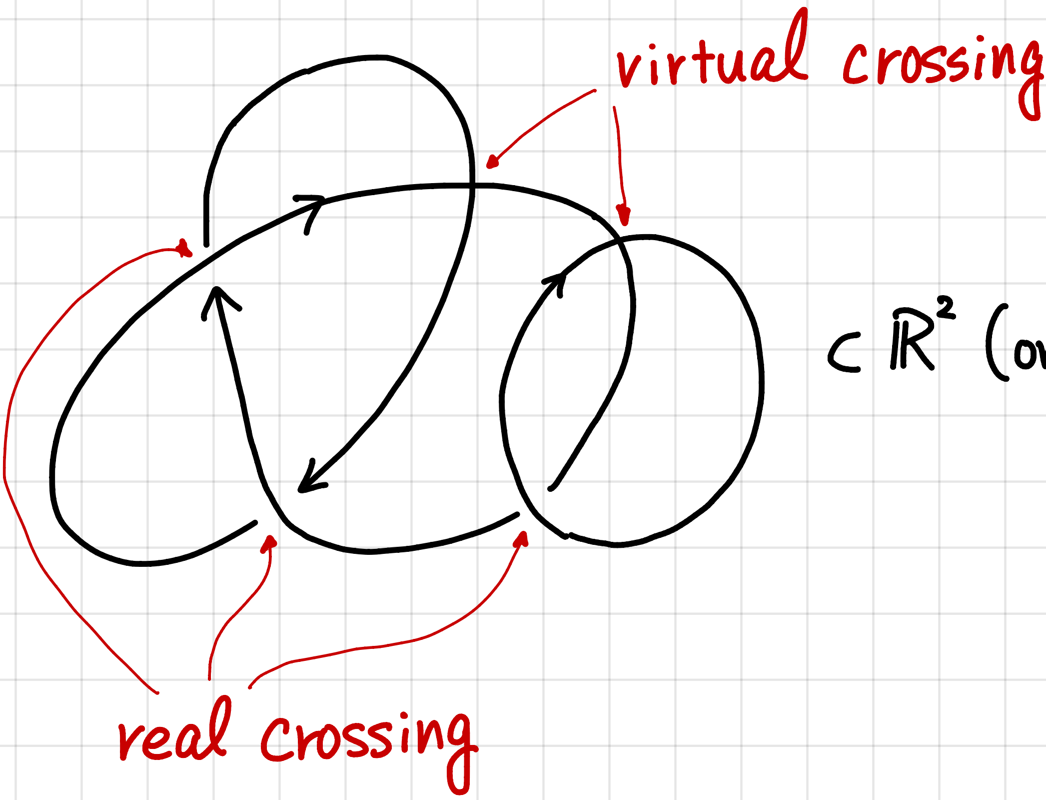
Abstract. We develop a diagrammatic calculus for welded knotted objects. We define Arrow presentations, which are essentially equivalent to Gauss diagrams but carry no sign on arrows, and more generally w-tree presentations, which can be seen as 'higher order Gauss diagrams'. We provide a complete set of moves for Arrow and w-tree presentations. This Arrow calculus is used to characterize finite type invariants of welded knots and long knots.

This is a joint work with Jean-Baptiste Meilhan (University of Grenoble Alpes).

§ Intro

Today,
we study only ori. diagram (1-dim in 2-dim)
(while we often omit ori)

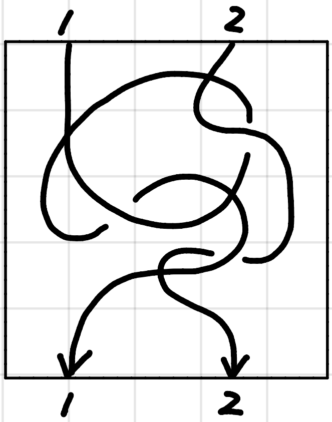
ori. link diagram

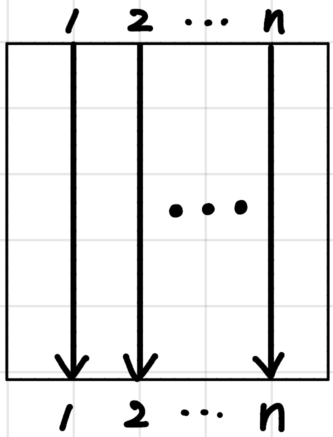


$\subset \mathbb{R}^2$ (or S^2)

$\bigcirc\ \bigcirc \cdots \bigcirc$: trivial

string link diagram

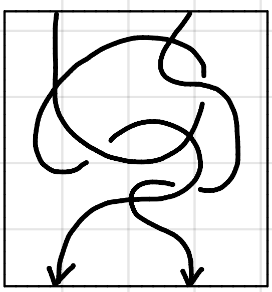
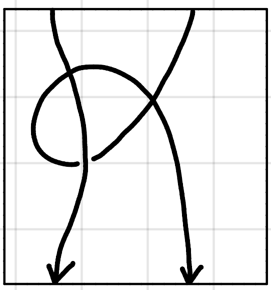
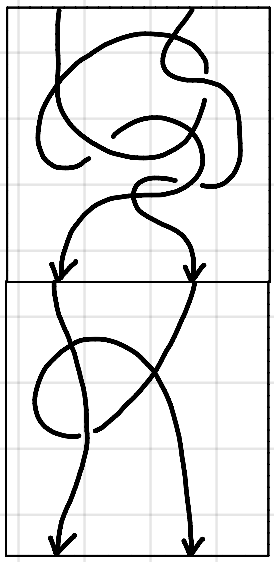




: trivial

(string) knot : (string) link of single curve

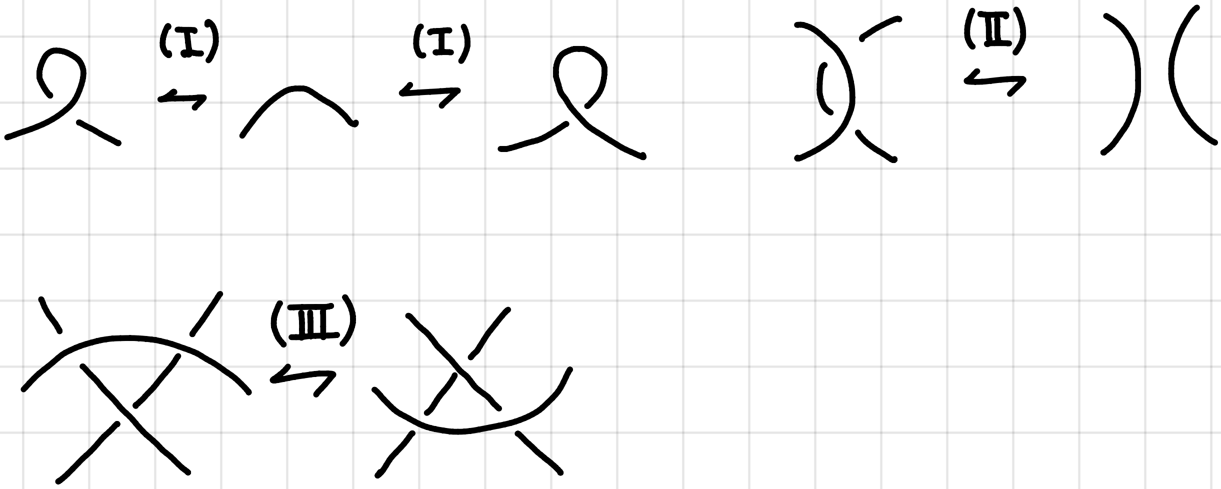
product of string link diagrams

 ×  := 

classical objects:

$$\{\text{diagrams with } \times\text{'s}\} / \text{R-moves}$$

R



Reidemeister's Th

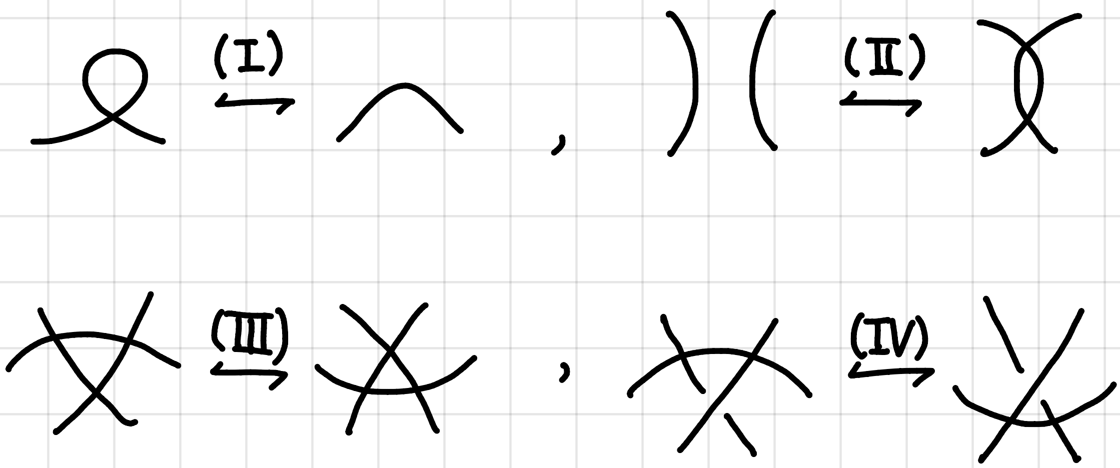
$$\{\text{links in } S^3\} / \text{amb. isotopy} \overset{=}{\longleftrightarrow} \{\text{diagrams with } \times\text{'s}\} / \text{R-moves}$$

embedded curves

virtual objects:

$\{$diagrams with $\times$'s & $\times$'s$\}$ / R-moves, vR-moves

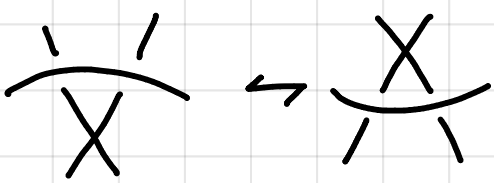
vR



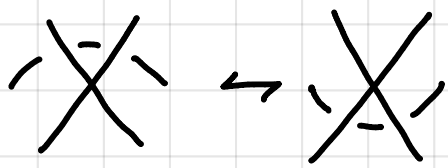
welded objects:

$\{$diagrams with $\times$'s & $\times$'s$\}$ / R-moves, vR-moves, OC

OC (over $\times$'s commute)



(UC (under $\times$'s commute)



"forbidden" move)

Rem. (Goussarov - Polyak - Viro '00)

$$\text{classical} \hookrightarrow \text{virtual}$$

$$\text{classical} \hookrightarrow \text{welded}$$

In this talk, our main object is welded.

i.e.,

diagram : diagram with &

$\sim$: equi up to R-moves, vR-moves, OC

Purpose

classical $\longleftrightarrow$ finite type inv (via $\to$)

Habiro's clasper theory

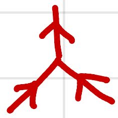
welded $\longleftrightarrow$ finite type inv (via $\to$)

Our "clasper theory"

§ w-tree & w-arrow

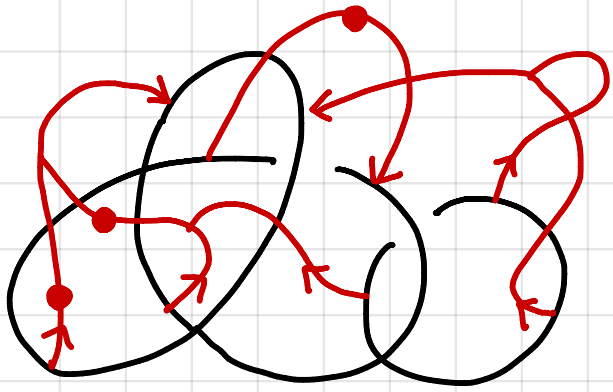
w-tree(s) for a diagram D :

immersed unitrivalent tree

with { ori. 

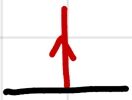
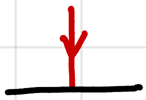
dot  (mod 2)

(i.e.  =)



s.t.

- {trivalent vertices} $\cap D = \phi$
- {univalent vertices} $\subset D \setminus \{$ ╳'s, ╳'s $\}$
- edge $\cap$ D, edge $\cap$ edge } virtual crossings

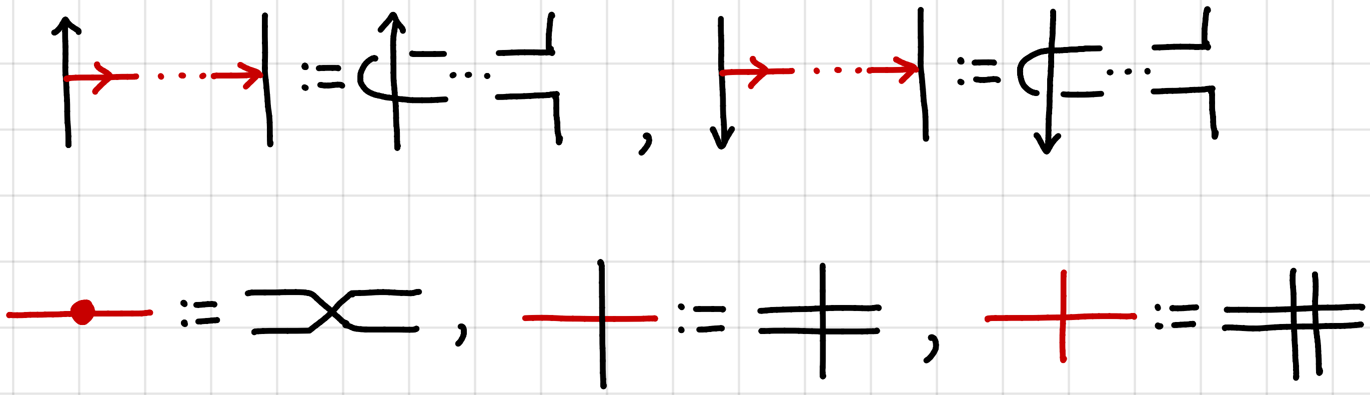
 : tail ,  : head

Note ∃! head (may omit to draw ori but head)

W_k-tree : w-tree with k tails

w-arrow : W_1-tree

surgery along w-arrow:



Arrow pres.

D : diagram, V: diagram without $\times$'s

A : w-arrows for V

(V, A) : arrow pres for D

$$\overset{\text{def}}{\iff} D \sim V_A$$

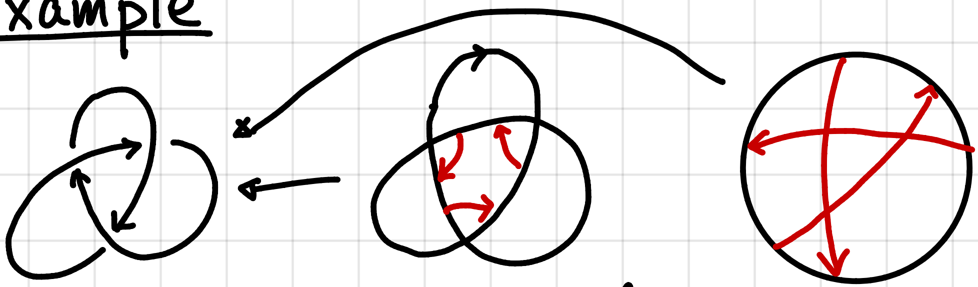
diagram obtained from V by surgery along A

$$(V, A) = (V', A') \overset{\text{def}}{\iff} V_A \sim V'_{A'}$$

Rem $\forall$ diagram has an arrow pres.

$D \longmapsto (V, A)$ canonical arrow pres

Example



D

canonical arrow pres for D

Question

$$\begin{array}{ccc} D & \longmapsto & (V, A) \\ \text{R, VR, OC} \wr & & \shortparallel \\ D' & \longmapsto & (V', A') \end{array}$$

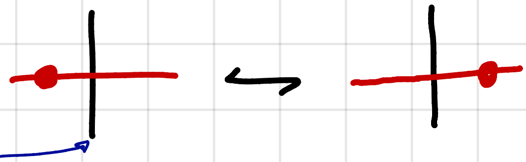
What kind of moves?

Th 1

$(V, A) = (V', A') \iff$ they are related by :

(0) (virtual isotopy)

vR-moves (I)~(III) &



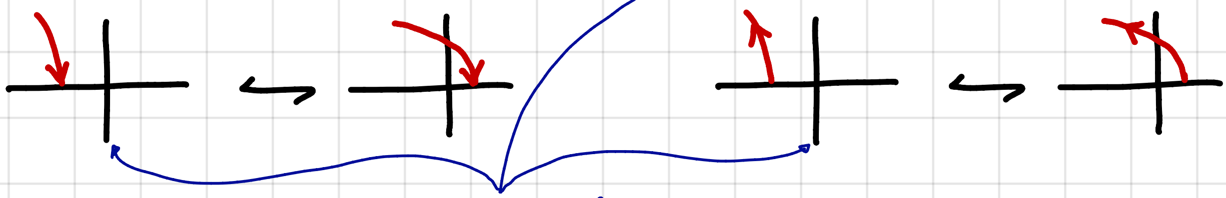
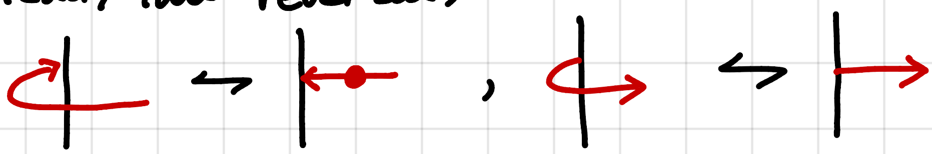
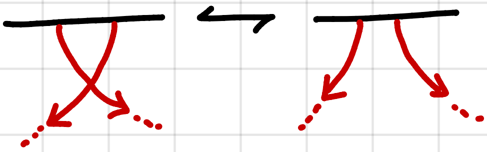


diagram or tree

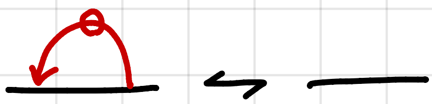
(1) (Head / tail reversal)

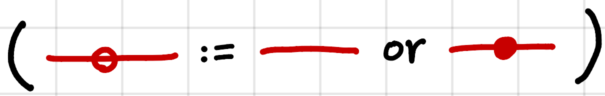


(2) (Tail exchange)

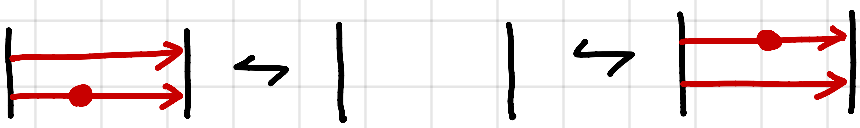


(3) (Isolated)

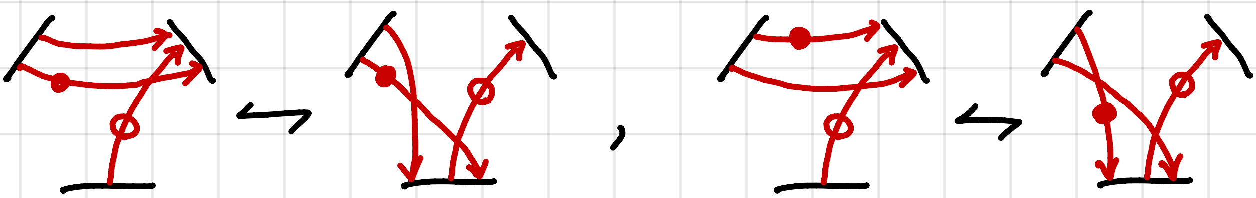




(4) (Inverse)



(5) (Slide)



Rem.

(0), (1) $\longleftrightarrow$ vR-moves, (2) $\longleftrightarrow$ OC

(3), (4), (5) $\longleftrightarrow$ R-moves (I), (II), (III) resp.

Rem (V, A) is similar to Gauss diagram

but { we do not need sign (—•— does not mean sign)

(V, A) is "topological" object

(that's why we need moves (0) & (1))

(Gauss diagram : "combinatrial" object which describes diagram.)

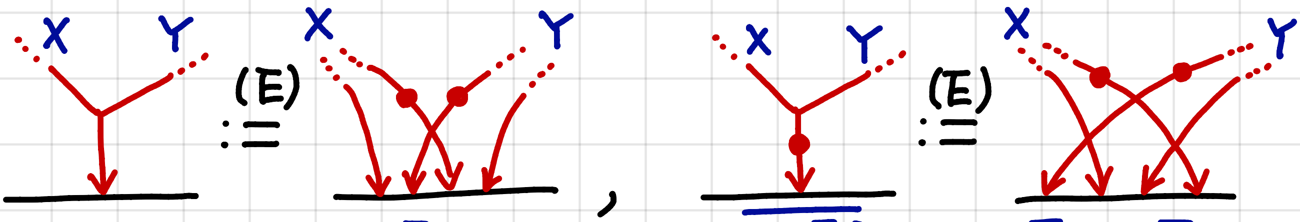
Our advantage

$$\begin{array}{ccc} \text{Arrow pres} & \xmapsto[\text{higher order}]{\text{generalize}} & \text{W-tree pres.} \\ \updownarrow & & \\ \text{Gauss diagram} & \longmapsto & \exists ?\ \text{higher order} \end{array}$$

will define soon

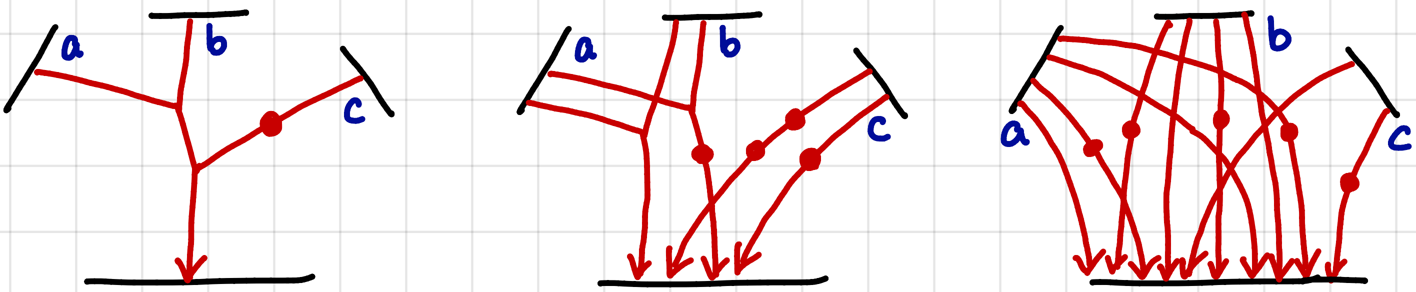
Surgery along w-tree

Extention (w-tree $\xmapsto{(E)}$ w-arrows)



$[X, \bar{Y}] = X\bar{Y}\bar{X}Y$, $\overline{[X, \bar{Y}]} = \bar{Y}XY\bar{X}$

Example



$$[[a\bar{b}]\bar{\bar{c}}] = [[a\bar{b}]c] = [a\bar{b}]\,c\,[\overline{a\bar{b}}]\,\bar{c} = a\bar{b}\bar{a}bc\bar{b}ab\bar{a}\bar{c}$$

(E)

surgery along w-tree

W : w-tree for a diagram D

$$D \cup W \overset{(E)}{\rightsquigarrow} D \cup \underbrace{A}_{\text{w-arrows}} \overset{\text{surgery}}{\rightsquigarrow} D_A$$

$D_W := D_A$: diagram obtained from D by surgery along W

Rem surgery along w-tree has "Brunnian property"

w-tree pres

D : diagram, V : diagram without $\times$'s

W : w-trees for V

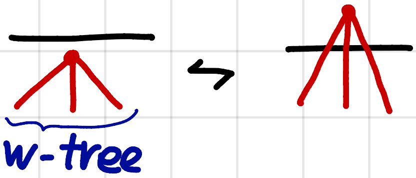
(V, W) : w-tree pres. for $D \overset{\text{def}}{\Longleftrightarrow} D \sim V_W$

$$(V, W) = (V', W') \overset{\text{def.}}{\Longleftrightarrow} V_W \sim V'_{W'}$$

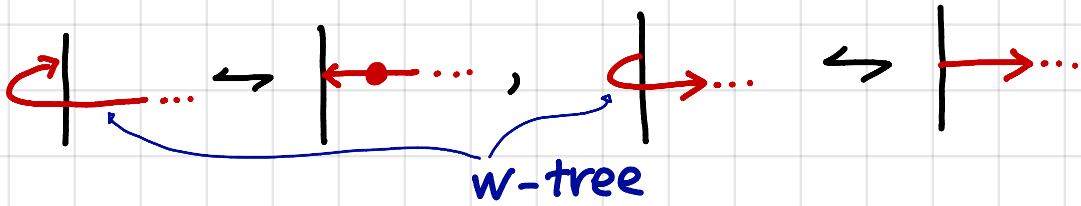
Th 2

$(V, W) = (V', W')$ if they related by moves (0) ~ (8):

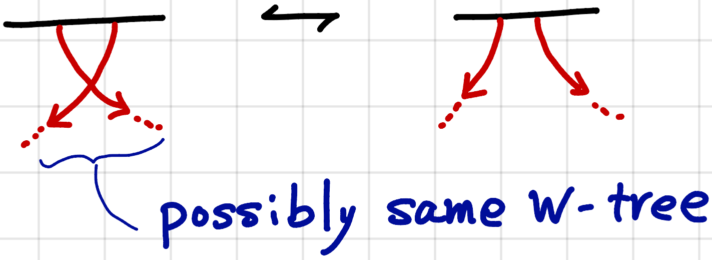
(0) virtual isotopy &



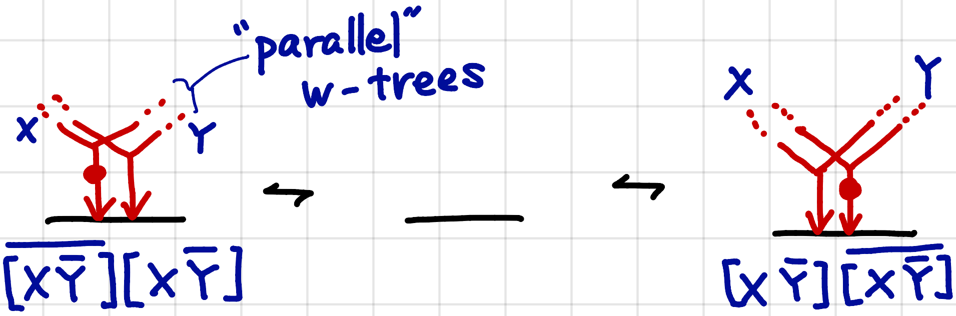
(1) (Head/tail reversal)



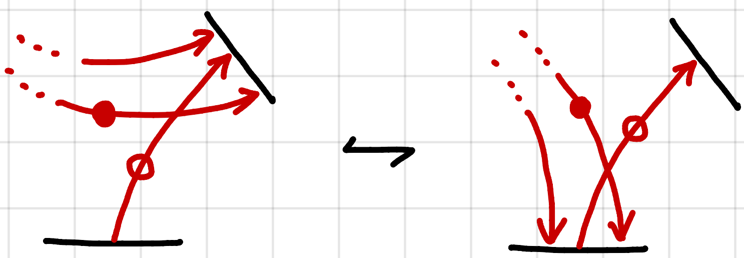
(2) (Tail exchange)



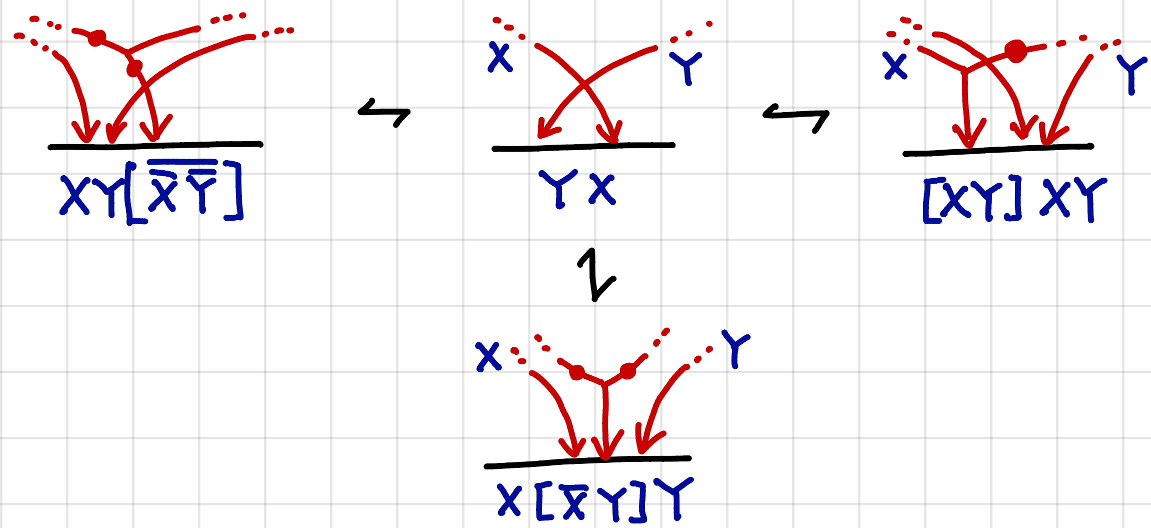
(3) (Inverse)



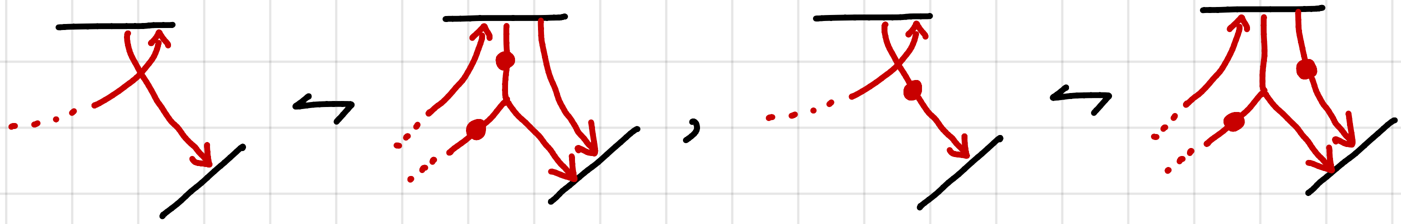
(4) (slide) (w-trees along w-arrow)



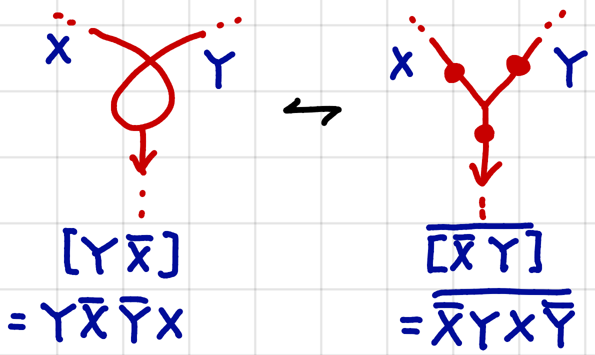
(5) (Head exchange)



(6) (Head-Tail exchange)
w-tree w-arrow

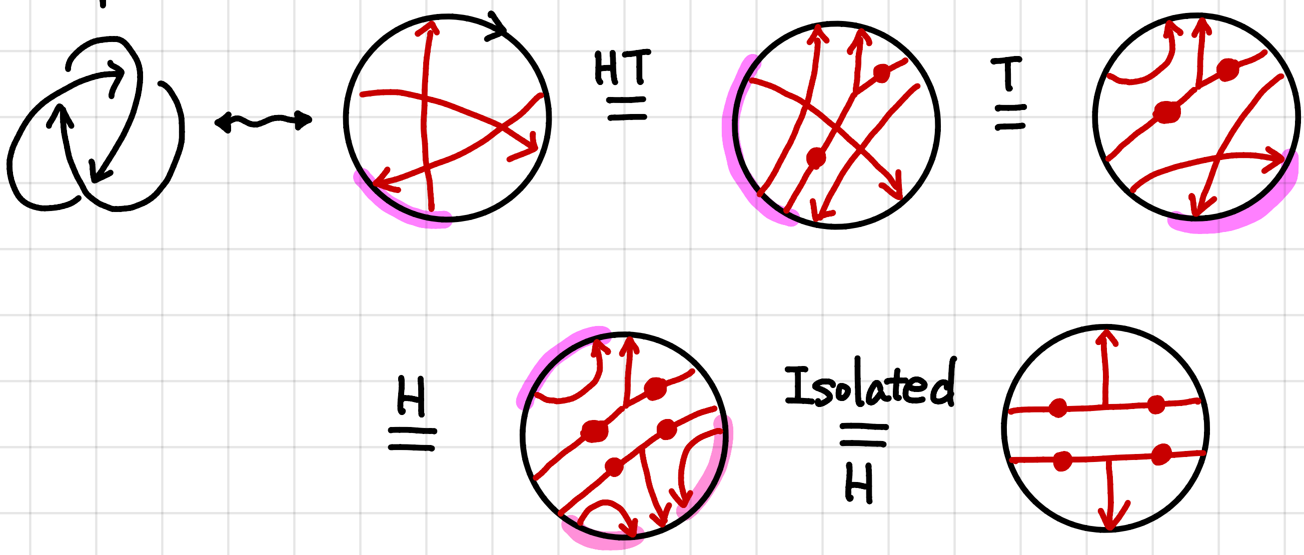


(7) (Antisymmetry)

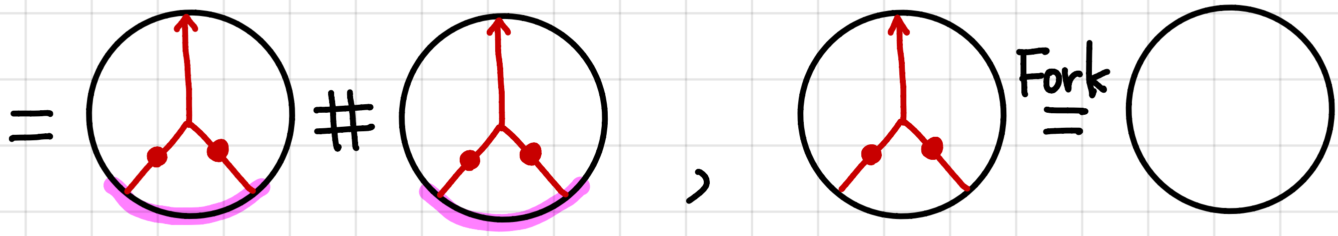


(8) (Fork)

Example



Note



trefoil (two bride knot) = connected sum of trivial knots

§ W_k-equi (welded version of Habiro's C_k-equi.)

D, D' : (string) link diagrams

$D \overset{k}{\sim} D'$: W_k-equi

$\overset{\text{def.}}{\Longleftrightarrow}$ ${}^{\exists} D = D_0, D_1, \ldots, D_m = D'$: seq. of diagrams

s.t. $\begin{cases} D_{i-1} \sim D_i, \text{ or} \\ D_{i-1} \xrightarrow[\text{w-tree of } \deg \geq k]{\text{surgery along}} D_i \end{cases}$

Rem

(1) $\overset{k+1}{\sim} \Rightarrow \overset{k}{\sim}$ (by def)

(2) $D \overset{k}{\sim} D' \Rightarrow {}^{\forall} \varphi$: finite type inv of $\deg \leq k-1$

$\varphi(D) = \varphi(D')$

(by Brunnian property)

Prop 3

$\{\text{welded string links}\}/\underset{k}{\sim}$: finitely generated group

Rem.

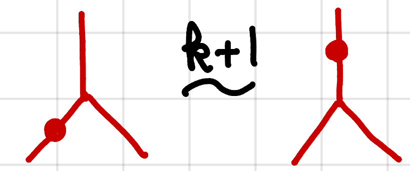
(1) $\{\text{welded links}\}/\underset{1}{\sim} = \{[\text{trivial link}]\}$

(2) $\{\text{welded } n\text{-comp. links}\}/\underset{2}{\sim}$

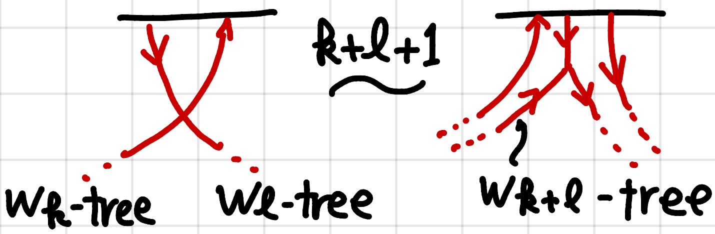
$=$ free abelian group of rank $n(n-1)$

Th 4

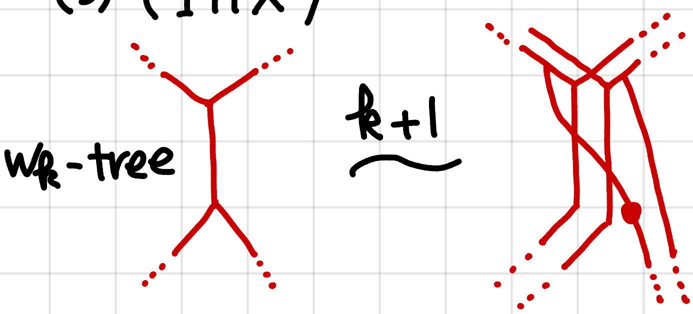
(1) (twist)



(2) (Head-Tail exchange)



(3) (IHX)



Th 5

$\forall$ welded knot $\overset{k}{\sim}$ $\bigcirc$ (for $\forall k \in \mathbb{N}$)

$\rightsquigarrow$ $\nexists$ nontrivial finite type welded knot inv.

Th 6

K, K' : welded string knots, (1)~(3) are equivalent

(1) $K \overset{k}{\sim} K'$

(2) $\forall \varphi$: finite type inv. of deg $\leq k-1$, $\varphi(K) = \varphi(K')$

(3) $\alpha_\ell(K) = \alpha_\ell(K')$ for $2 \leq \forall \ell \leq k-1$

where α_ℓ : coef of $(t-1)^\ell$ in the Tayler exp.
of the normalized Alex poly

(Habiro - Kanenobu - Shima '99)

Cor 7

$\{$welded string knots$\}/\overset{k}{\sim}$: free abelian group
of rank $k-2$

(For $k = 1, 2$, this group is trivial)

<u>Rem.</u> By using Yajima-Satoh's tube map,
Th 6 induce

(1) (Habiro-Shima '01)
Finite type inv. of ribbon 2-knots are determined by the Alex. poly

(2) (Watanabe '06)
ribbon 2-knots are RC_k-equi.
$\Longleftrightarrow$
their finite type inv. of $\deg \leq k-1$ coincide