A formula for the action of Dehn twists on the HOMFLY-PT type skein algebra and its application

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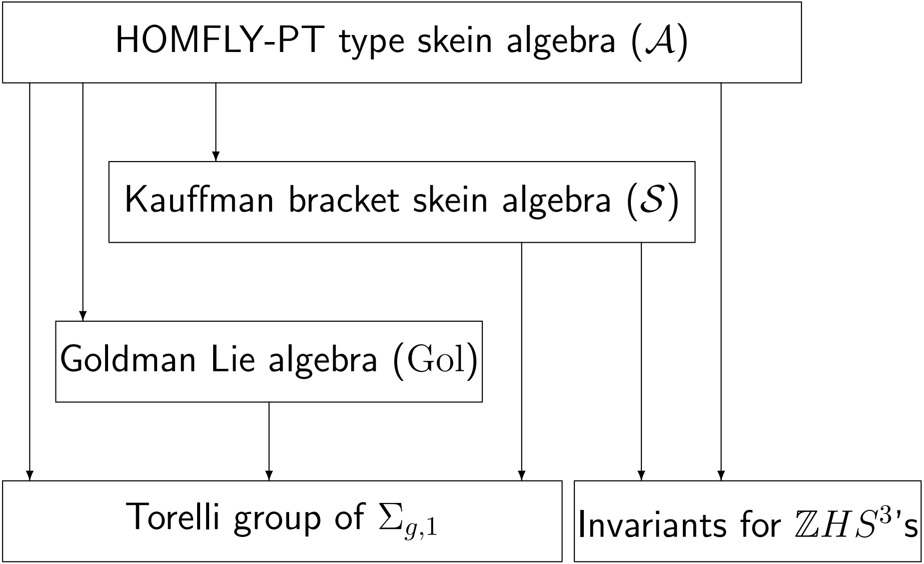
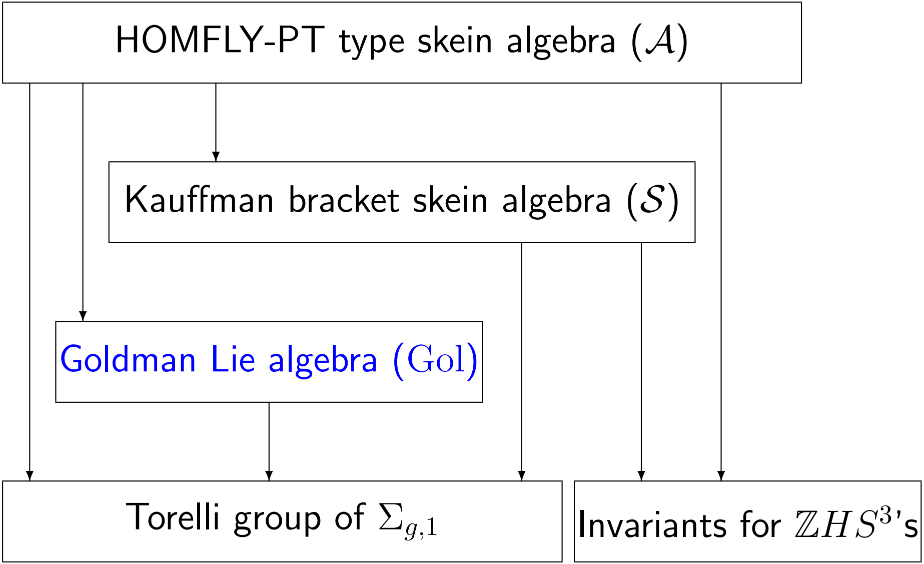
Outline of this talk




HOMFLY-PT type skein algebra ($\mathcal{A}$)
Kauffman bracket skein algebra ($\mathcal{S}$)
Goldman Lie algebra (Gol)
Torelli group of $\Sigma_{g,1}$
Invariants for $\mathbb{Z}HS^3$'s

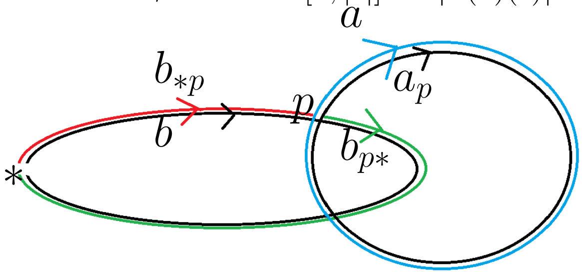
Goldman Lie algebra 1/4

Let Σ be a compact connected oriented surface with $* \in \partial\Sigma \neq \emptyset$ and $\hat{\pi}(\Sigma)$ be the set of the conjugacy classes of $\pi_1(\Sigma, *)$. We denote by $|\cdot|$ the quotient map $\mathbb{Q}\pi_1(\Sigma, *) \to \mathbb{Q}\hat{\pi}(\Sigma)$.
For $a \in \hat{\pi}(\Sigma)$ and $b \in \pi_1(\Sigma, *)$ in general position, we define

$$\sigma(a)(b) \stackrel{\text{def.}}{=} \sum_{p \in a \cap b} \varepsilon(p, a, b) b_{*p} a_p b_{p*},$$

where $\varepsilon(p, a, b)$ is the local intersection number of a and b at p.

Furthermore, we define $[a, |b|] \stackrel{\text{def.}}{=} |\sigma(a)(b)|$.



Goldman Lie algebra 2/4

Goldman proves that $(\mathbb{Q}\hat{\pi}(\Sigma), [\ \ ,\ \ ])$ is a Lie algebra. We call this Lie algebra the Goldman Lie algebra of Σ. Furthermore, Kawazumi and Kuno prove that $\mathbb{Q}\pi_1(\Sigma, *)$ is a Lie module of Goldman Lie algebra $(\mathbb{Q}\hat{\pi}(\Sigma), [\ \ ,\ \ ])$. We define the augmentation map

$$\epsilon_{\mathrm{Gol}} : \mathbb{Q}\pi_1(\Sigma, *) \to \mathbb{Q}, x \in \pi_1(\Sigma, *) \mapsto 1.$$

Then, we have

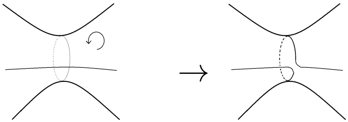
$$[|(\ker \epsilon_{\mathrm{Gol}})^n|, |(\ker \epsilon_{\mathrm{Gol}})^m|] \subset |(\ker \epsilon_{\mathrm{Gol}})^{n+m-2}|,$$
$$\sigma(|(\ker \epsilon_{\mathrm{Gol}})^n|)((\ker \epsilon_{\mathrm{Gol}})^m) \subset (\ker \epsilon_{\mathrm{Gol}})^{n+m-2},$$
$$\cap_{i=0}^{\infty} |(\ker \epsilon_{\mathrm{Gol}})^i| = \{0\}, \cap_{i=0}^{\infty} (\ker \epsilon_{\mathrm{Gol}})^i = \{0\}$$

We denote

$$\widehat{\mathbb{Q}\pi_1}(\Sigma, *) \stackrel{\mathrm{def.}}{=} \varprojlim_{i \to \infty} \mathbb{Q}\pi_1(\Sigma, *)/(\ker \epsilon_{\mathrm{Gol}})^i,$$
$$\widehat{\mathbb{Q}\hat{\pi}}(\Sigma) \stackrel{\mathrm{def.}}{=} \varprojlim_{i \to \infty} \mathbb{Q}\hat{\pi}(\Sigma)/|(\ker \epsilon_{\mathrm{Gol}})^i|.$$

Goldman Lie algebra 3/4

We denote the mapping class group of Σ by $\mathcal{M}(\Sigma) \stackrel{\text{def.}}{=} \mathrm{Diff}^+(\Sigma, \partial\Sigma)/$ (isotopy rel. $\partial\Sigma$ pointwise). For a simple closed curve c in Σ, we denote the (right hand) Dehn twist along c by $t_c \in \mathcal{M}(\Sigma)$.



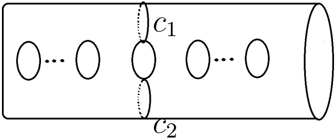
Theorem (Kawazumi-Kuno, Massuyeau-Turaev)

We define $L_{\mathrm{Gol}}(c) = \frac{1}{2}|(\log(\gamma))^2|$ *where* γ *is an element of* $\pi_1(\Sigma)$ *such that* $|\gamma| = c$. *Then, we have*

$$t_c(\cdot) = \exp(\sigma(L_{\mathrm{Gol}}(c)))(\cdot) \in \mathrm{Aut}(\widehat{\mathbb{Q}\pi_1}(\Sigma, *)),$$

where $\exp(\sigma(L_{\mathrm{Gol}}(c)))(\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!}(\sigma(L_{\mathrm{Gol}}(c)))^i(\cdot)$.

Goldman Lie algebra 4/4



For $x, y \in |(\ker \epsilon_{\mathrm{Gol}})^3|$, we define

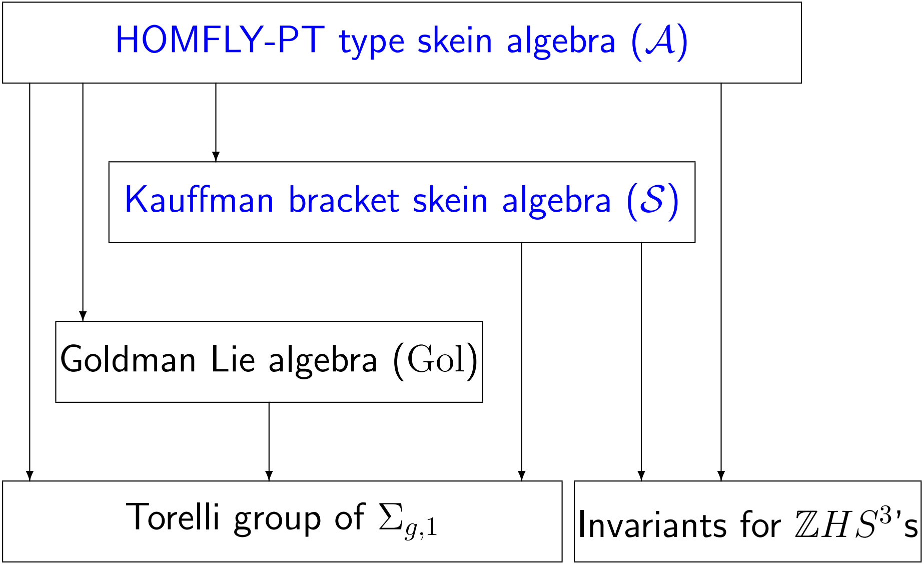
$$\mathrm{bch}(x,y) \stackrel{\mathrm{def.}}{=} \log(\exp(x)\exp(y))$$
$$= x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) \cdots .$$

Using the Baker Campbell Hausdorff series bch, we can consider $|(\ker \epsilon_{\mathrm{Gol}})^3|$ as a group.

Let $\Sigma_{g,1}$ be a connected compact oriented surface of genus g with connected nonempty boundary. We know that the Torelli group $\mathcal{I}(\Sigma_{g,1}) \stackrel{\mathrm{def.}}{=} \ker(\mathcal{M}(\Sigma_{g,1}) \to \mathrm{Aut}(H_1(\Sigma_{g,1},\mathbb{Z})))$ is generated by $\{t_{c_1}{t_{c_2}}^{-1} | c_1, c_2 : \mathrm{BP}\}$.

Theorem (Kawazumi-Kuno)

The group homomorphism $\zeta_{\mathrm{Gol}} : \mathcal{I}(\Sigma_{g,1}) \to (|(\ker \epsilon)^3|, \mathrm{bch})$ *defined by* $t_{c_1}{t_{c_2}}^{-1} \mapsto L_{\mathrm{Gol}}(c_1) - L_{\mathrm{Gol}}(c_2)$ *for BP* (c_1, c_2) *is well-defined and injective.*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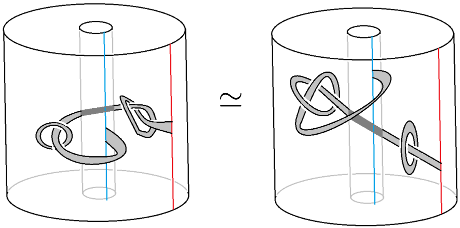
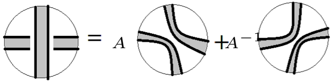
Skein algebras in $\Sigma \times [0,1]$ 1/3

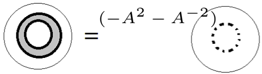
Let Σ be a compact connected oriented surface and J a finite subset of $\partial\Sigma$. We denote by $\mathcal{T}(\Sigma, J)$ the set of framed unoriented tangles in $\Sigma \times [0,1]$ with base point set J.



the skein relation



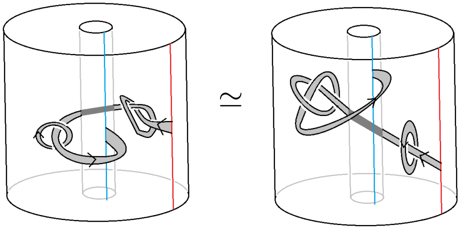
the trivial knot relation

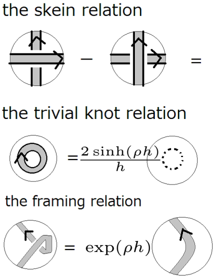


Let $\mathcal{S}(\Sigma, J)$ be the quotient of $\mathbb{Q}[A^{\pm 1}]\mathcal{T}(\Sigma, J)$ by the relation. We simply denote $\mathcal{S}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{S}(\Sigma, \emptyset)$. We call $\mathcal{S}(\Sigma)$ the Kauffman bracket skein algebra of Σ.

Skein algebras in $\Sigma \times [0,1]$ 2/3

Let Σ be a compact connected oriented surface and J^- and J^+ finite subsets of $\partial\Sigma$ satisfying $J^- \cap J^+ = \emptyset$. We denote by $\mathcal{T}^+(\Sigma, J^-, J^+)$ the set of framed oriented tangles in $\Sigma \times [0,1]$ with start point set and end point set J^- and J^+.



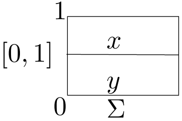


Let $\mathcal{A}(\Sigma, J^-, J^+)$ be the quotient of $\mathbb{Q}[\rho][[h]]\mathcal{T}^+(\Sigma, J^-, J^+)$ by the relation. We remark $\mathbb{Q}[\rho][[h]] \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty} \mathbb{Q}[\rho, h]/(h^i)$. We simply denote $\mathcal{A}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{A}(\Sigma, \emptyset, \emptyset)$. We call $\mathcal{A}(\Sigma)$ the HOMFLY-PT type skein algebra of Σ.

Skein algebras in $\Sigma \times [0,1]$ 3/3

We define the products of $\mathcal{S}(\Sigma)$ and $\mathcal{A}(\Sigma)$ and the right actions and the left actions of $\mathcal{S}(\Sigma)$ and $\mathcal{A}(\Sigma)$ on $\mathcal{S}(\Sigma, J)$ and $\mathcal{A}(\Sigma, J^-, J^+)$ by the following.

$xy \stackrel{\text{def.}}{=}$

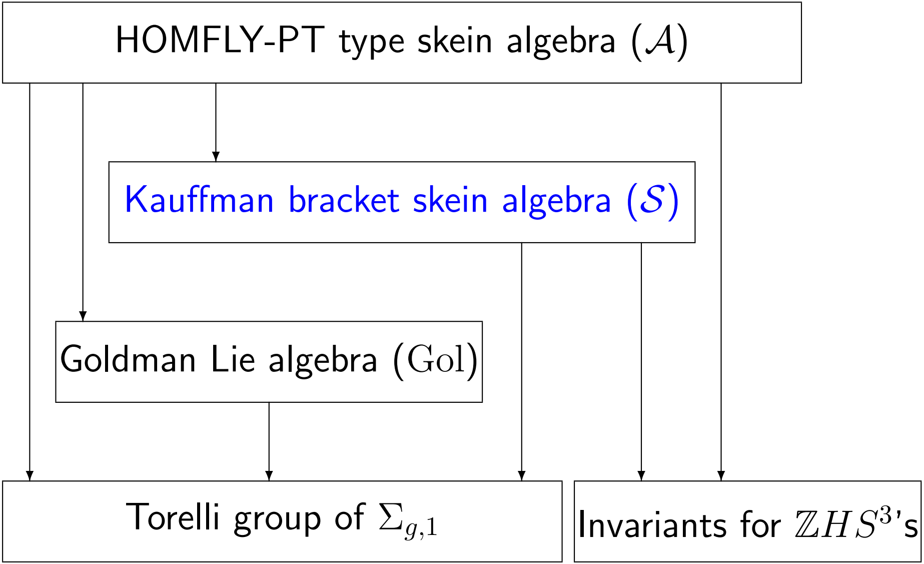


for $(x, y) \in \mathcal{S}(\Sigma) \times \mathcal{S}(\Sigma)$, $\mathcal{S}(\Sigma, J) \times \mathcal{S}(\Sigma)$, $\mathcal{S}(\Sigma) \times \mathcal{S}(\Sigma, J)$, $\mathcal{A}(\Sigma) \times \mathcal{A}(\Sigma)$, $\mathcal{A}(\Sigma, J^-, J^+) \times \mathcal{A}(\Sigma)$ or $\mathcal{A}(\Sigma) \times \mathcal{A}(\Sigma, J^-, J^+)$

Proposition

We obtain the following isomorphisms

$$\phi_{\mathcal{S}} : \mathcal{S}(D^2) \to \mathbb{Q}[A^{\pm 1}],$$
$$\phi_{\mathcal{A}} : \mathcal{A}(D^2) \to \mathbb{Q}[\rho][[h]].$$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Kauffman bracket skein algebra 1/5

We define the bracket of $\mathcal{S}(\Sigma)$ by

$$[x, y] \stackrel{\text{def.}}{=} \frac{1}{-A + A^{-1}}(xy - yx)$$

for x and $y \in \mathcal{S}(\Sigma)$. Then $(\mathcal{S}(\Sigma), [\ \ ,\ \ ])$ is a Lie algebra. We define the Lie action σ of $\mathcal{S}(\Sigma)$ on $\mathcal{S}(\Sigma, J)$ by

$$\sigma(x)(z) \stackrel{\text{def.}}{=} \frac{1}{-A + A^{-1}}(xz - zx)$$

for $x \in \mathcal{S}(\Sigma)$ and $z \in \mathcal{S}(\Sigma, J)$. Then $\mathcal{S}(\Sigma, J)$ is a $(\mathcal{S}(\Sigma), [\ \ ,\ \ ])$-module.
The augmentation map $\epsilon_{\mathcal{S}}$ is defined by $\epsilon_{\mathcal{S}}(L) = (-2)^{\sharp \pi_0(L)}$ and $\epsilon_{\mathcal{S}}(A) = -1$. Using this augmentation map, we will define a filtration of $\mathcal{S}(\Sigma)$.

Kauffman bracket skein algebra 2/5

We define a filtration $\{F^n\mathcal{S}(\Sigma)\}_{n\geq 0}$ satisfying the following.

Proposition

We obtain

$$F^{2n}\mathcal{S}(\Sigma) = (\ker \epsilon_{\mathcal{S}})^n$$
$$F^n\mathcal{S}(\Sigma)F^m\mathcal{S}(\Sigma) \subset F^{n+m}\mathcal{S}(\Sigma),$$
$$[F^n\mathcal{S}(\Sigma), F^m\mathcal{S}(\Sigma)] \subset F^{n+m-2}\mathcal{S}(\Sigma) \quad \text{for} \quad n, m \geq 0.$$

Proposition

Let χ *be an embedding* $\Sigma \times [0,1] \to \Sigma' \times [0,1]$. *Then* $\chi(F^n\mathcal{S}(\Sigma)) \subset F^n\mathcal{S}(\Sigma')$ *for any* n.

Kauffman bracket skein algebra 3/5

$H \stackrel{\text{def.}}{=} H_1(\Sigma, \mathbb{Q})$.

Proposition

If $\partial\Sigma \neq \emptyset$, there are $\mathbb{Q}$-linear isomorphisms

$$\lambda_2 : S^2(H) \oplus \mathbb{Q} \to F^2\mathcal{S}(\Sigma)/F^3\mathcal{S}(\Sigma),$$
$$\lambda_3(= \lambda) : \wedge^3 H \to F^3\mathcal{S}(\Sigma)/F^4\mathcal{S}(\Sigma),$$
$$\lambda_4 : S^2(S^2(H)) \oplus S^2(H) \oplus \mathbb{Q} \to F^4\mathcal{S}(\Sigma)/F^5\mathcal{S}(\Sigma).$$

Proposition

If $\partial\Sigma \neq \emptyset$, we obtain

$$\cap_{i=0}^{\infty} F^i\mathcal{S}(\Sigma) = \{0\}, \cap_{i=0}^{\infty} F^i\mathcal{S}(\Sigma)\mathcal{S}(\Sigma, J) = \{0\}.$$

Kauffman bracket skein algebra 4/5

We denote

$$\widehat{\mathcal{S}}(\Sigma) \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty} \mathcal{S}(\Sigma)/F^i\mathcal{S}(\Sigma),$$
$$\widehat{\mathcal{S}}(\Sigma, J) \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty} \mathcal{S}(\Sigma, J)/F^i\mathcal{S}(\Sigma)\mathcal{S}(\Sigma, J).$$

Theorem (T.)

For a simple closed curve c,

$$L_{\mathcal{S}}(c) \stackrel{\text{def.}}{=} \frac{-A + A^{-1}}{4\log(-A)}(\operatorname{arccosh}(\frac{-c}{2}))^2 - (-A + A^{-1})\log(-A).$$

Then we have

$$t_c(\cdot) = \exp(\sigma(L_{\mathcal{S}}(c)))(\cdot) \in \operatorname{Aut}(\widehat{\mathcal{S}}(\Sigma)).$$

Kauffman bracekt skein algebra 5/5

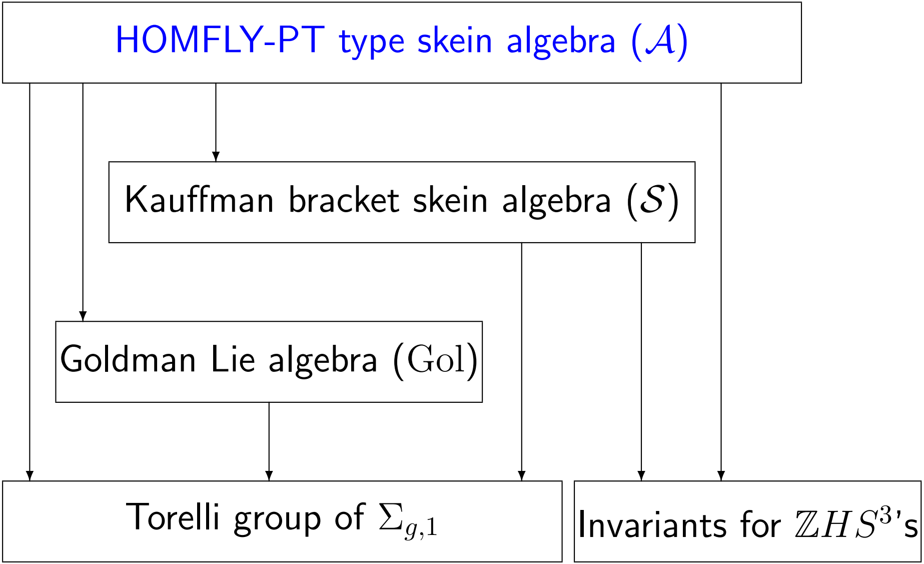
Using the Baker Campbell Hausdorff series bch, we can consider $F^3\widehat{\mathcal{S}}(\Sigma)$ as a group.

Theorem (T.)

The group homomorphism $\zeta_{\mathcal{S}} : \mathcal{I}(\Sigma_{g,1}) \to (F^3\widehat{\mathcal{S}}(\Sigma_{g,1}), \mathrm{bch})$ *defined by* $t_{c_1}{t_{c_2}}^{-1} \mapsto L_{\mathcal{S}}(c_1) - L_{\mathcal{S}}(c_2)$ *for BP* (c_1, c_2) *is well-defined and injective, where*

$$L_{\mathcal{S}}(c) \stackrel{\text{def.}}{=} \frac{-A + A^{-1}}{4\log(-A)}(\operatorname{arccosh}(\frac{-c}{2}))^2 - (-A + A^{-1})\log(-A).$$

We remark that, for null-homologous simple closed curve c, $\zeta_{\mathcal{S}}(t_c) = L_{\mathcal{S}}(c)$.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HOMFLY-PT type skein algebra 1/4

We define a Lie bracket $[\ ,\ ]$ of $\mathcal{A}(\Sigma)$ and a Lie action σ of $\mathcal{A}(\Sigma)$ on $\mathcal{A}(\Sigma, J^-, J^+)$ satisfying $h[x,y] = xy - yx, h\sigma(x)(z) = xz - zx$. There exsit filtrations $\{F^n\mathcal{A}(\Sigma)\}_{n\geq 0}$ and $\{F^n\mathcal{A}(\Sigma, J^-, J^+)\}_{n\geq 0}$ of $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma, J)$ satisfying the following.

Proposition

The product and the Lie bracket of $\mathcal{A}(\Sigma)$ and the right action, the left action and the Lie action of $\mathcal{A}(\Sigma)$ on $\mathcal{A}(\Sigma, J^-, J^+)$ are continuous in the topology induced by the filtrations. In particular,

$$[F^n\mathcal{A}(\Sigma), F^m\mathcal{A}(\Sigma)] \subset F^{n+m-2}\mathcal{A}(\Sigma).$$

Proposition

Let χ be an embedding $(\Sigma\times[0,1], J^-\times[0,1], J^+\times[0,1]) \to (\Sigma'\times[0,1], J'^-\times[0,1], J'^+\times[0.1])$. Then $\chi(F^n\mathcal{A}(\Sigma, J^-, J^+)) \subset F^n\mathcal{A}(\Sigma', J'^-, J^+)$ for any n.

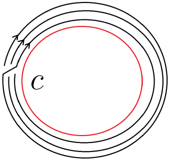
HOMFLY-PT type skein algebra 2/4

We denote

$$\widehat{\mathcal{A}}(\Sigma) \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty} \mathcal{A}(\Sigma)/F^i\widehat{\mathcal{S}}(\Sigma),$$

$$\widehat{\mathcal{A}}(\Sigma, J) \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty} \mathcal{A}(\Sigma, J^-, J^+)/F^i\mathcal{A}(\Sigma, J^-, J^+).$$

In $\widehat{\mathcal{A}}(\Sigma)$, we also define $L_{\mathcal{A}}(c)$ for s.c.c. c.

$l_n(c) \stackrel{\text{def.}}{=}$  for $n = 3$.

$$m_n(c) \stackrel{\text{def.}}{=} \sum_{1\leq j\leq n} \frac{(-h)^{j-1}}{j} \sum_{i_1+\cdots i_j=n} l_{i_1}(c)l_{i_2}(c)\cdots l_{i_j}(c), m_0(c) \stackrel{\text{def.}}{=} 2\rho$$

$$(m-1)_n(c) \stackrel{\text{def.}}{=} \sum_{0\leq i\leq n} \frac{n!}{i!(n-i)!}(-1)^{n-i}m_i(c) \in F^n\widehat{\mathcal{A}}(\Sigma).$$

HOMFLY-PT type skein algebra 3/4

$$L_{\mathcal{A}}(c) \stackrel{\text{def.}}{=} (\frac{h/2}{\operatorname{arcsinh}(h/2)})^2 \sum_{i\geq 2} u_i(m-1)_i(c) - \frac{1}{3}\rho^3 h^2,$$

where $\frac{1}{2}(\log(X))^2 = \sum_{i\geq 2} u_i (X-1)^i$.

Theorem (T.)

For any simple closed curve c, then we have

$$t_c(\cdot) = \exp(\sigma(L_{\mathcal{A}}(c)))(\cdot) \in \operatorname{Aut}(\widehat{\mathcal{A}}(\Sigma, J^-, J^+)).$$

Theorem (T.)

The group homomorphism $\zeta_{\mathcal{A}} : \mathcal{I}(\Sigma_{g,1}) \to (F^3\widehat{\mathcal{A}}(\Sigma_{g,1}), \mathrm{bch})$ defined by $t_{c_1} {t_{c_2}}^{-1} \mapsto L_{\mathcal{A}}(c_1) - L_{\mathcal{A}}(c_2)$ for BP (c_1, c_2) is well-defined and injective. Furthermore, for null-homologous s.c.c. c, $\zeta_{\mathcal{A}}(t_c) = L_{\mathcal{A}}(c)$.

HOMFLY-PT type skein algebra 4/4

There exists a surjective Lie algebra homomorphism

$$\psi_{\mathcal{AG}} : \mathcal{A}(\Sigma) \to \mathbb{Q}\hat{\pi}(\Sigma).$$

For any n, the Lie algebra homomorphism satisfies

$$\psi_{\mathcal{AG}}(F^n\mathcal{A}(\Sigma)) = |(\ker \epsilon_{\mathrm{Gol}})^n|.$$

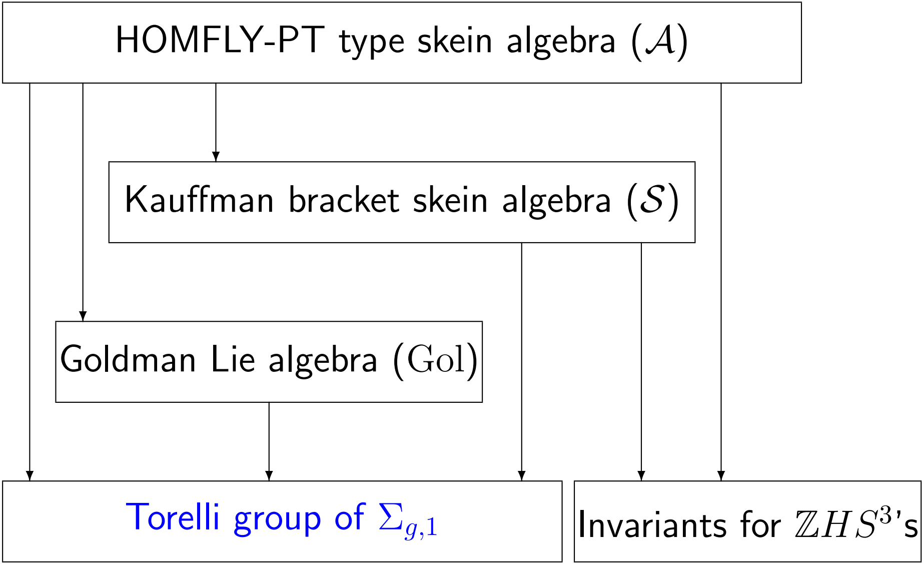
Let $\mathcal{A}^0(\Sigma)$ be the $\mathbb{Q}[\rho][[h]]$ submodule of $\mathcal{A}(\Sigma)$ generated by links whose homology class is 0. There exists a Lie algebra homomorphism

$$\psi_{\mathcal{AS}} : \mathcal{A}^0(\Sigma) \to \mathcal{S}(\Sigma).$$

For any n, the Lie algebra homomorphism satisfies

$$\psi_{\mathcal{AS}}(F^n\mathcal{A}(\Sigma) \cap \mathcal{A}^0(\Sigma)) \subset F^n\mathcal{S}(\Sigma).$$

We denote $\widehat{\mathcal{A}^0}(\Sigma) \stackrel{\mathrm{def.}}{=} \varprojlim_{i\to\infty}\mathcal{A}^0(\Sigma)/F^i\mathcal{A}(\Sigma) \cap \mathcal{A}^0(\Sigma)$.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Some filtrations of $\mathcal{I}(\Sigma_{g,1})$ 1/3

The filtrations

$$\begin{aligned}
\mathcal{I}(\Sigma_{g,1}) = & \mathcal{I}_g = \mathcal{I}_g(1) \supset \mathcal{I}_g(2) \cdots (\text{lower central series}), \\
& \mathcal{I}_g = \mathcal{M}_g(1) \supset \mathcal{M}_g(2) \cdots (\text{Johnson filtration}), \\
& \mathcal{I}_g = \mathcal{M}_g^{\mathrm{S}}(1) \supset \mathcal{M}_g^{\mathrm{S}}(2) \cdots (\text{Kauffman filtration}), \\
& \mathcal{I}_g = \mathcal{M}_g^{\mathrm{A}}(1) \supset \mathcal{M}_g^{\mathcal{A}}(3) \cdots (\text{HOMFLY}-\text{PT filtration}),
\end{aligned}$$

are defined by

$$\begin{aligned}
& \mathcal{I}_g(k) = [\mathcal{I}_g(k-1), \mathcal{I}_g], \\
& \mathcal{M}_g(k) = \zeta_{\mathrm{Gol}}^{-1}(|(\ker \epsilon_{\mathrm{Gol}})^{k+2}|), \\
& \mathcal{M}_g^{\mathcal{S}}(k) = \zeta_{\mathcal{S}}^{-1}(F^{k+2}\widehat{\mathcal{S}}(\Sigma_{g,1})), \mathcal{M}_g^{\mathcal{A}}(k) = \zeta_{\mathcal{A}}^{-1}(F^{k+2}\widehat{\mathcal{A}}(\Sigma_{g,1})).
\end{aligned}$$

We have $\mathcal{I}_g(k) \subset \mathcal{M}_g(k), \mathcal{M}_g^{\mathcal{S}}(k), \mathcal{M}_g^{\mathcal{A}}(k)$.

Some filtrations of $\mathcal{I}(\Sigma_{g,1})$ 2/3

Theorem (Johnson)

$\mathcal{K}_g \stackrel{\text{def.}}{=} \mathcal{M}_g(2)$ *is generated by Dehn twists along null homologous simple closed curves.*

Theorem (T.)

$\mathcal{K}_g \stackrel{\text{def.}}{=} \mathcal{M}_g(2) = \mathcal{M}_g^{\mathcal{S}}(2) = \mathcal{M}_g^{\mathcal{A}}(2)$.

Corollary

$\zeta_{\mathcal{A}}(\mathcal{M}_g^{\mathcal{A}}(2)) \subset \widehat{\mathcal{A}^0}(\Sigma_{g,1})$

Using $\psi_{\mathcal{A}G}$ and $\psi_{\mathcal{A}\mathcal{S}}$, we obtain the following.

Theorem (T.)

$\mathcal{M}_g(k) \supset \mathcal{M}_g^{\mathcal{A}}(k) \supset \mathcal{I}_g(k), \mathcal{M}_g^{\mathcal{S}}(k) \supset \mathcal{M}_g^{\mathcal{A}}(k) \supset \mathcal{I}_g(k)$ *for any* k.

Some filtrations of $\mathcal{I}(\Sigma_{g,1})$ 3/3

Let $\tau_2 : \mathcal{K}_g \to S^2(\wedge^2 H)$ be the second Johnson homomorphism. We remark that $\ker \tau_2 = \mathcal{M}_g(3)$.

Theorem (T.)

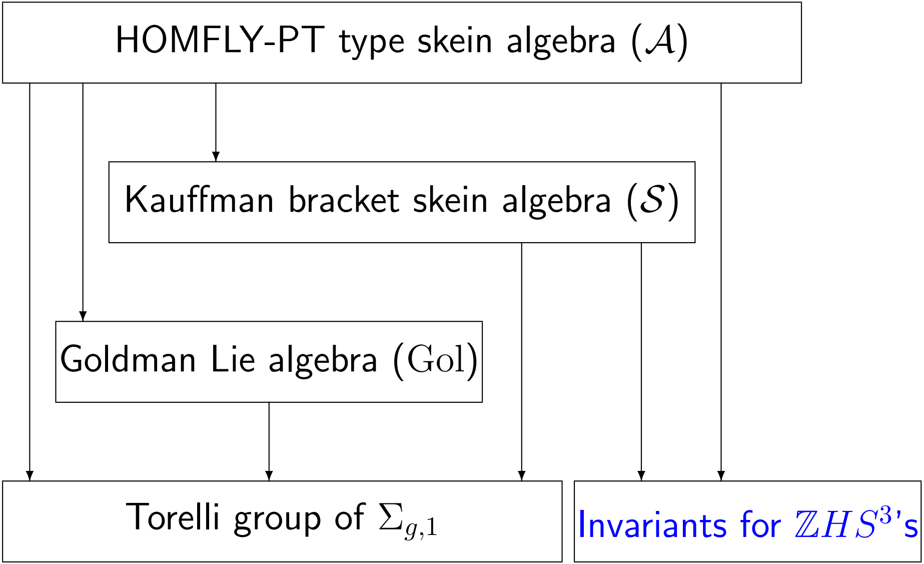
Let $\upsilon_1 \oplus \upsilon_2 \oplus \upsilon_3$ be the group homomorphism

$$\mathcal{K}_g \stackrel{\zeta_{\mathcal{S}}}{\to} F^4\widehat{\mathcal{S}}(\Sigma_{g,1}) \twoheadrightarrow F^4\widehat{\mathcal{S}}(\Sigma_{g,1})/F^5\widehat{\mathcal{S}}(\Sigma_{g,1}) \stackrel{\lambda^{-1}}{\to} S^2(S^2(H)) \oplus S^2(H) \oplus \mathbb{Q}.$$

Then $\upsilon_1 = \nu \circ \tau_2$, $\upsilon_2 = 0$ and υ_3 is the core of the Casson invariant defined by Morita. Here
$\nu((a_1 \wedge a_2) \cdot (a_3 \wedge a_4)) \stackrel{\text{def.}}{=} (a_1 \cdot a_3) \cdot (a_2 \cdot a_4) - (a_1 \cdot a_4) \cdot (a_2 \cdot a_3)$.

Corollary

$\mathcal{M}_g^{\mathcal{S}}(3) \not\supset \mathcal{M}_g(3)$ *and* $\mathcal{M}_g^{\mathcal{S}}(3) \not\subset \mathcal{M}_g(3)$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Invariants for $\mathbb{Z}HS^3$'s

Some invariants for $\mathbb{Z}HS^3$s 1/6

Let $\mathfrak{M}(3)$ be the set of closed oriented 3-manifolds and $\mathfrak{H}(3) \subset \mathfrak{M}(3)$ be the set of integral homology 3-spheres ($\mathbb{Z}HS^3$s).
Fix a Heegaard decomposition of S^3 by

$$S^3 = H_g^+ \cup_\iota H_g^-,$$

where H_g^+ and H_g^- are handle bodies and ι is a diffeomorphism $\partial H_g^+ \to \partial H_g^-$. We consider $\Sigma_{g,1}$ as a submanifold of ∂H_g^+. For $\xi \in \mathcal{M}(\Sigma_{g,1})$, we denote

$$M(\xi) \stackrel{\text{def.}}{=} H_g^+ \cup_{\iota \circ \xi} H_g^-.$$

Theorem (Reidemeister-Singer)

$(\coprod_g \mathcal{M}(\Sigma_{g,1}))/\text{R. S. stabilization} = \mathfrak{M}(3)$
$(\coprod_g \mathcal{I}(\Sigma_{g,1}))/\text{R. S. stabilization} = \mathfrak{H}(3)$

Some invariants for $\mathbb{Z}HS^3$s 2/6

Let $e : \Sigma_{g,1} \times [0,1] \to S^3$ be the (orientation preserving) collar neighborhood of $\Sigma_{g,1} \subset H_g^+ \subset S^3$. This embedding e induces $\mathbb{Q}[[A+1]]$-module homomorphism $e : \widehat{\mathcal{S}}(\Sigma_{g,1}) \to \mathbb{Q}[[A+1]]$ and $\mathbb{Q}[\rho][[h]]$-module homomorphism $e : \widehat{\mathcal{A}}(\Sigma_{g,1}) \to \mathbb{Q}[\rho][[h]]$.

Theorem (T.)

The map

$$Z_{\mathcal{S}} : \mathcal{I}(\Sigma_{g,1}) \to \mathbb{Q}[[A+1]], \xi \mapsto \sum_{i=0}^{\infty} \frac{1}{(-A+A^{-1})^i i!} e((\zeta_{\mathcal{S}}(\xi))^i)$$

induces

$$z_{\mathcal{S}} : \mathfrak{H}(3) \to \mathbb{Q}[[A+1]], M(\xi) \mapsto Z_{\mathcal{S}}(\xi).$$

In other words, for $M \in \mathfrak{H}(3)$,
$z_{\mathcal{S}}(M) = 1 + z_{\mathcal{S}}^1(M)(A+1) + z_{\mathcal{S}}^2(M)(A+1)^2 + \cdots$ *is an invariant for M.*

Some invariants for $\mathbb{Z}HS^3$s 3/6

Proposition

For $M \in \mathfrak{H}(3)$, $z_{\mathcal{S}}(M) = 1 + z_{\mathcal{S}}^1(M)(A+1) + \cdots + z_{\mathcal{S}}^n(M)(A+1)^n$ $\mathrm{mod}\ ((A+1)^{n+1})$ *is a finite type invariants of order* n*.*

Corollary

For $M \in \mathfrak{H}(3)$, $-\frac{1}{24} z_{\mathcal{S}}^1(M)$ *is the Casson invariant for* M*.*

Some invariants for $\mathbb{Z}HS^3$s 4/6

Theorem (Morita 1989)

There exists a group homomorphism $d : \mathcal{K}_g \to \mathbb{Q}$ satisfying

1. $d(\xi'\xi\xi'^{-1}) = d(\xi)$ *for* $\xi \in \mathcal{K}_g$ *and* $\xi' \in \mathcal{M}(\Sigma_{g,1})$.
2. *There exists a* $\mathbb{Q}$*-linear map* $d' : S^2(\wedge^2 H) \to \mathbb{Q}$ *such that the Casson invariant for* $M(\xi)$ *equals* $d(\xi) + d'(\tau_2(\xi))$.

We call this group homomorphism d *the core of the Casson invariant.*

Let $v_1 \oplus v_2 \oplus v_3$ be the group homomorphism

$$\mathcal{K}_g \stackrel{\zeta_{\mathcal{S}}}{\to} F^4\widehat{\mathcal{S}}(\Sigma_{g,1}) \twoheadrightarrow F^4\widehat{\mathcal{S}}(\Sigma_{g,1})/F^5\widehat{\mathcal{S}}(\Sigma_{g,1}) \stackrel{\lambda^{-1}}{\to} S^2(S^2(H)) \oplus S^2(H) \oplus \mathbb{Q}.$$

Since, the map

$\mathcal{K}_g \to \mathcal{M}_g^{\mathcal{S}}(3)/\mathcal{M}_g^{\mathcal{S}}(4) \stackrel{\zeta_{\mathcal{S}}}{\to} F^3\widehat{\mathcal{S}}(\Sigma_{g,1})/F^4\widehat{\mathcal{S}}(\Sigma_{g,1}) \stackrel{z^1_{\mathcal{S}}(M(\cdot))}{\to} \mathbb{Q}$ is a group homomorphism, v_3 is the core of the Casson invariant.

Some invariants for $\mathbb{Z}HS^3$s 5/6

Theorem (T.)

The map

$$Z_{\mathcal{A}} : \mathcal{I}(\Sigma_{g,1}) \to \mathbb{Q}[\rho][[h]], \xi \mapsto \sum_{i=0}^{\infty} \frac{1}{h^i i!} e((\zeta_{\mathcal{A}}(\xi))^i)$$

induces

$$z_{\mathcal{A}} : \mathfrak{H}(3) \to \mathbb{Q}[\rho][[h]], M(\xi) \mapsto Z_{\mathcal{A}}(\xi).$$

Proposition

For $M \in \mathfrak{H}(3)$, $(z_{\mathcal{A}}(M))_{|\exp(\rho h)=A^4, h=-A^2+A^{-2}} = z_{\mathcal{S}}(M)$.

Proposition

For $M \in \mathfrak{H}(3)$, $z_{\mathcal{A}}(M) \mod (h^{n+1})$ is a finite type invariants of order n.

Some invariants for $\mathbb{Z}HS^3$s 6/6

Question

For $M \in \mathfrak{H}(3)$, is $(z_{\mathcal{S}}(M))_{|A^4=q} \in \mathbb{Q}[[q-1]]$ the Ohtsuki series?

Proposition

Let M_1 be the Poincaré homology 3-sphere. Then $(z_{\mathcal{S}}(M_1))_{|A^4=q} =$ (Ohtsuki series for M_1) $\mod ((q-1)^{13})$.

Question

For $M \in \mathfrak{H}(3)$, is $(z_{\mathcal{A}}(M))_{|\exp(\rho h)=q^{(n+1)/2}, h=-q^{1/4}+q^{1/4}} \in \mathbb{Q}[[q-1]]$ the invariant defined by Habiro and Le via the quantum group of sl_n?