Non equivalent coverings between fibred hyperbolic manifolds

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Topology and Geometry of Low-dimensional Manifolds

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The aim of the talk will be to answer the following question in the positive and discuss some consequences in dimension 3:

Question

Are there two conjugate pseudo-Anosov maps φ_1, φ_2 defined on a closed, connected, oriented surface S and two non-equivalent finite coverings $p_1, p_2 : \hat{S} \to S$ such that a lift f_1 of φ_1 via p_1 coincides with a lift f_2 of φ_2 via p_2?

Outline of the talk

- Some motivation
- An application and some remarks
- Main idea providing the positive answer
- First part of the proof: construction of surface covers
- Second part: definition of pseudo-Anosov maps
- More remarks
- Extension to the branched case and an application

Question

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Let L be a codimension-2 submanifold of B. Assuming there is a ramified covering map $E \to B$ with branch locus L, under which hypotheses is the covering determined by E, B, and L and to what extent?

Examples (what is trivially known)

If *E* and *B* be are surfaces, one can use the Euler characteristic to establish whether *E* can cover *B*. If it does and the Euler characteristic is not 0, then the order of the cover is fixed (but not the cover!).

Of course, a torus covers itself in many different ways for every possible order.

Motivation

Examples (what is trivially known)

If *E* and *B* be are surfaces and *L* a finite set of points in *B*, one can use the Riemann-Hurwitz formula to establish whether *E* can branch-cover *B*.

Note that in this case, even in the hyperbolic setting, *E*, *B*, and *L* are not sufficient to fix the order of the cover, unless all points have the same order of ramification.

Examples (what is known)

If *E* and *B* be are 3-manifolds, the Euler characteristic becomes useless. If the manifolds are hyperbolic, their volumes give necessary conditions for the existence of a cover, whose order would then be determined. More generally one has:

Theorem (Wang, Wu, Yu)

The order of the cover is determined by *E* and *B* provided they do not admit a geometry of type $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$, $\mathbf{E}^3$, *Nil*, or *Sol*.

Examples (what is known)

For ramified coverings of 3-manifolds one has:

Theorem (Salgueiro)

Assume that E and B-L are irreducible with incompressible boundary and with JSJ-decomposition of B-L dual to a tree. Then there is at most one prime q for which E is a q-fold covering of B with branch locus L.

Moreover, in the hyperbolic setting it suffices to require that the cover is strongly cyclic instead of prime order.

Question

Is it possible to strengthen the results in the hyperbolic setting?

An application and some remarks

Let φ_1, φ_2 be two conjugate pseudo-Anosov maps defined on a closed, connected, oriented surface *S.*

Let $p_1, p_2 : \hat{S} \rightarrow S$ be two non-equivalent finite coverings such that a lift f_1 of φ_1 via p_1 coincides with a lift f_2 of φ_2 via p_2.

Proposition (Los, P., Salgueiro)

Let *E* be the mapping torus of $f_1=f_2$ and *B* that of φ_1 (this is also the mapping torus of φ_2 since the psuedo-Anosov are conjugate). *E* and *B* are hyperbolic and *E* admits two inequivalent covers over *B,* induced by p_1 and p_2, which preserve the natural fibrations by construction.

Remarks

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One can adapt the construction to show that for each k there are hyperbolic manifolds that cover k distinct manifolds, each in two non equivalent ways.

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By taking *m*th powers of the different pseudo-Anosov maps, one gets infinitely many instances of this phenomenon where all manifolds are commensurable.

One can adapt the construction to show that for each *k* there are hyperbolic manifolds that cover *k* distinct manifolds, each in two non equivalent ways.

In the end we will adapt this construction to branched covers.

Main idea providing the positive answer

Theorem (Los, P., Salgueiro)

Let S be a closed, connected, oriented surface of genus >2.

There are two conjugate pseudo-Anosov maps φ_1, φ_2 defined on S and two non-equivalent finite coverings $p_1, p_2 : \hat{S} \to S$ such that a lift f_1 of φ_1 via p_1 coincides with a lift f_2 of φ_2 via p_2.

Moreover there are infinitely many such pairs φ_1, φ_2, in the sense that if φ'_1, φ'_2 is another pair, no power of φ_i is conjugate to a power of φ'_j.

Main idea providing the positive answer

Proof (main idea)

Assume we have the following commuting diagram of finite regular coverings

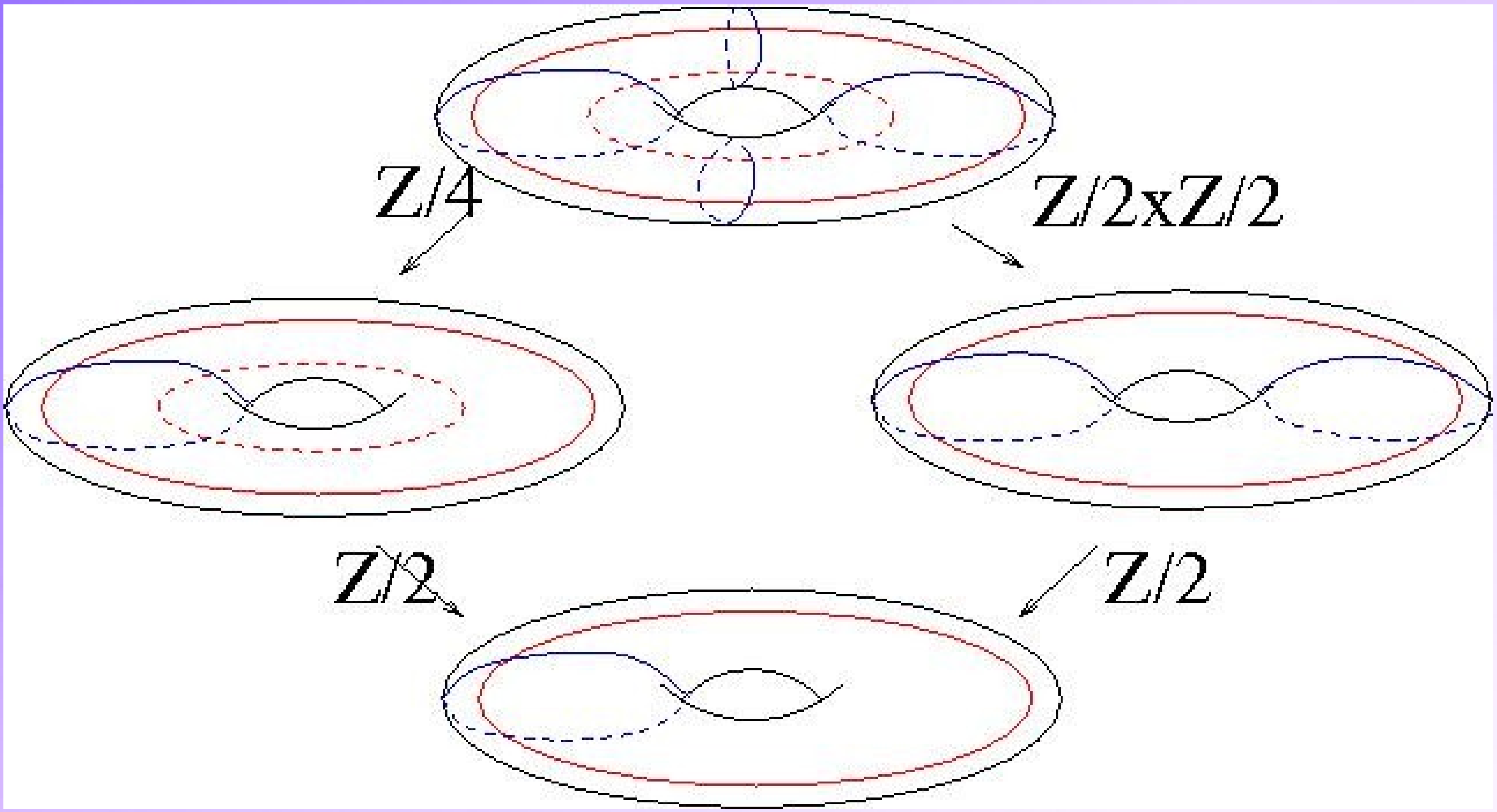
$$\begin{array}{ccc} & \hat{S} & \\ p_1 \swarrow & & \searrow p_2 \\ S & & S \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & \overline{S} & \end{array}$$

where p_1 and p_2 are not equivalent but π_1 and π_2 are. Up to taking powers, a well-chosen map on $\overline{S}$ lifts as we want.

First part of the proof: construction of surface covers

Let $n \geq 2$ and consider the group **Z**/n^2 x **Z**/n acting on a torus by translations permuting n^2 meridians and n longitudes.

This gives the following system of covers (here n=2):

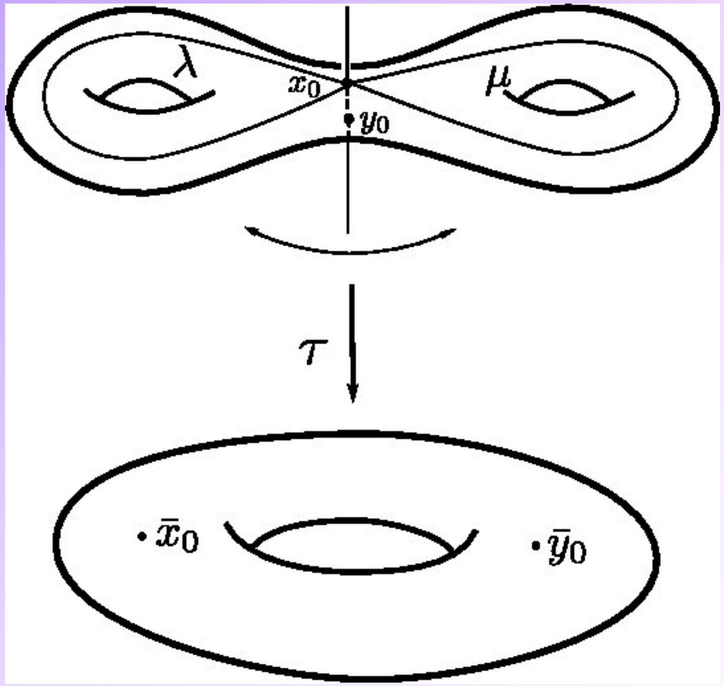


First part of the proof: construction of surface covers

To get a system of surface coverings, take the boundaries of small regular neighbourhoods of the meridians and longitudes considered. Note that the resulting S has genus $n+1\geq 3$.

The two bottom coverings are equivalent because there is a homeomorphism of the torus that exchanges meridians and longitudes.

On the surface:

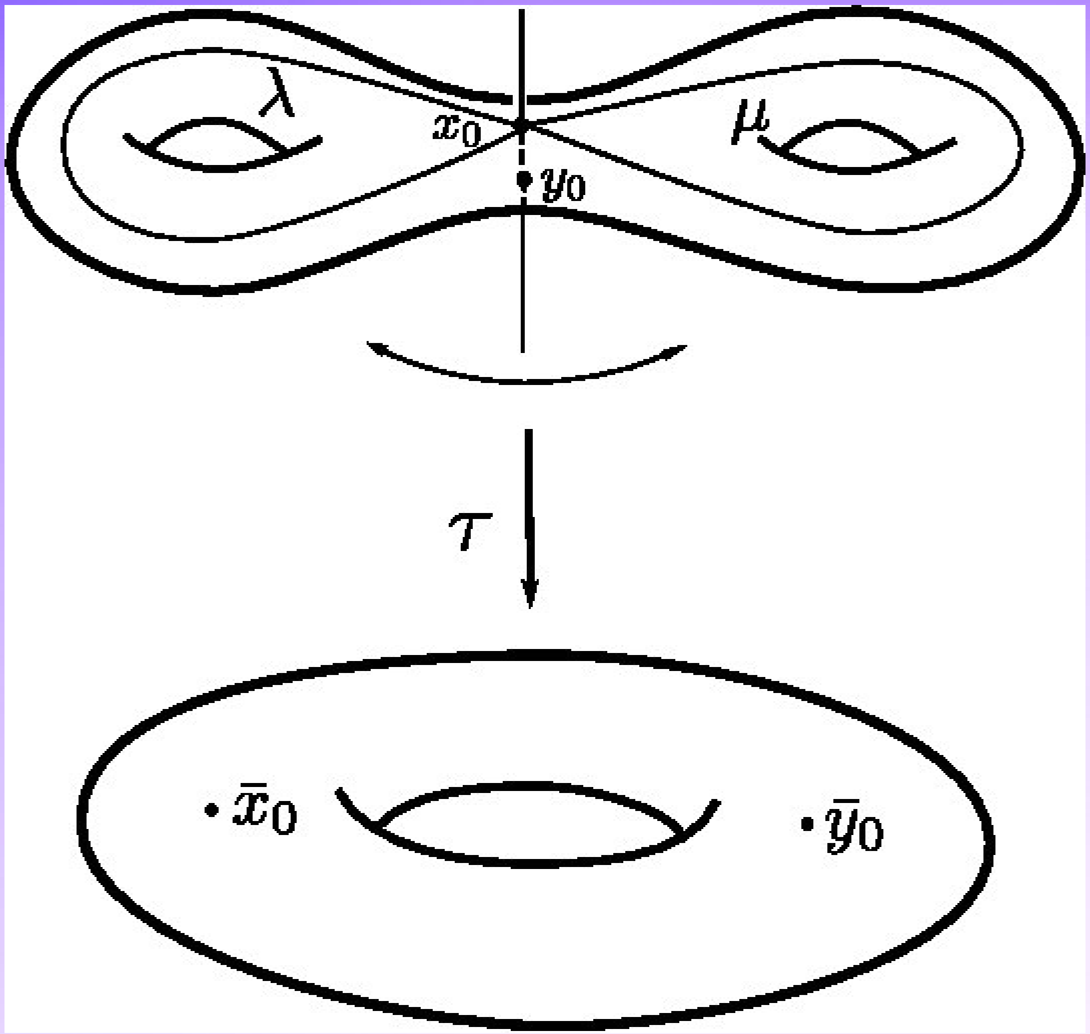


Second part: definition of pseudo-Anosov maps

Choose an Anosov map with two fixed points, identified with $\bar{x}_0$ and $\bar{y}_0$.

Up to taking a sufficiently large power, the Anosov map can be lifted to all surfaces (so that the lifts fix point-wise the fibre of x_0).

The lift to $\bar{S}$ commutes with τ and so the lifts to S are conjugate by construction.



All lifts are pseudo-Anosov maps.

More remarks

Remark

The existence of infinitely many pairs φ_1, φ_2 comes from the existence of infinitely many conjugacy classes of Anosov maps.

This is not sufficient to conclude that there are infinitely many manifolds behaving like this, because a hyperbolic manifold can fibre in infinitely many ways.

More remarks

Remark

To show the existence of infinitely many manifolds, one can use either the fact that we have infinitely many distinct groups acting ($\mathbf{Z}/n$), or take covers using powers of the maps.

In principle, all these examples are commensurable. A different construction, using arithmetic manifolds, due to Reid and Salgueiro ensures the existence of infinitely many commensurability classes, too.

More remarks

Remark

The same construction can be carried out with other groups of the form **Z**/*m* x **Z**/*n*, provided there is a prime *p* dividing both *n* and *m* and such that the Sylow *p*-subgroups of **Z**/*m* and **Z**/*n* are not isomorphic.

For $m=n^2$ and $n=p_1...p_k$, a product of *k* distinct primes, one can consider the covers associated to the subgroups **Z**/*m* x **Z**/(n/p_i) and **Z**/(m/p_i) x **Z**/*n*. This gives a manifold that covers *k* other manifolds, each in two different ways.

More remarks

Remark and open question

All hyperbolic manifolds are fibred and all covers respect the given fibration. Will it be possible to find manifolds for which one can carry out the construction for two different fibrations at the same time?

Extension to the branched case and an application

We now want to consider a system of branched covers:

$$S_g$$

$$\downarrow$$

$$S_5(m,m,n,n)$$

$$\swarrow \qquad\qquad \searrow$$

$$S_2(2,2,2m,2m,n) \qquad\qquad S_2(2,2,m,2n,2n)$$

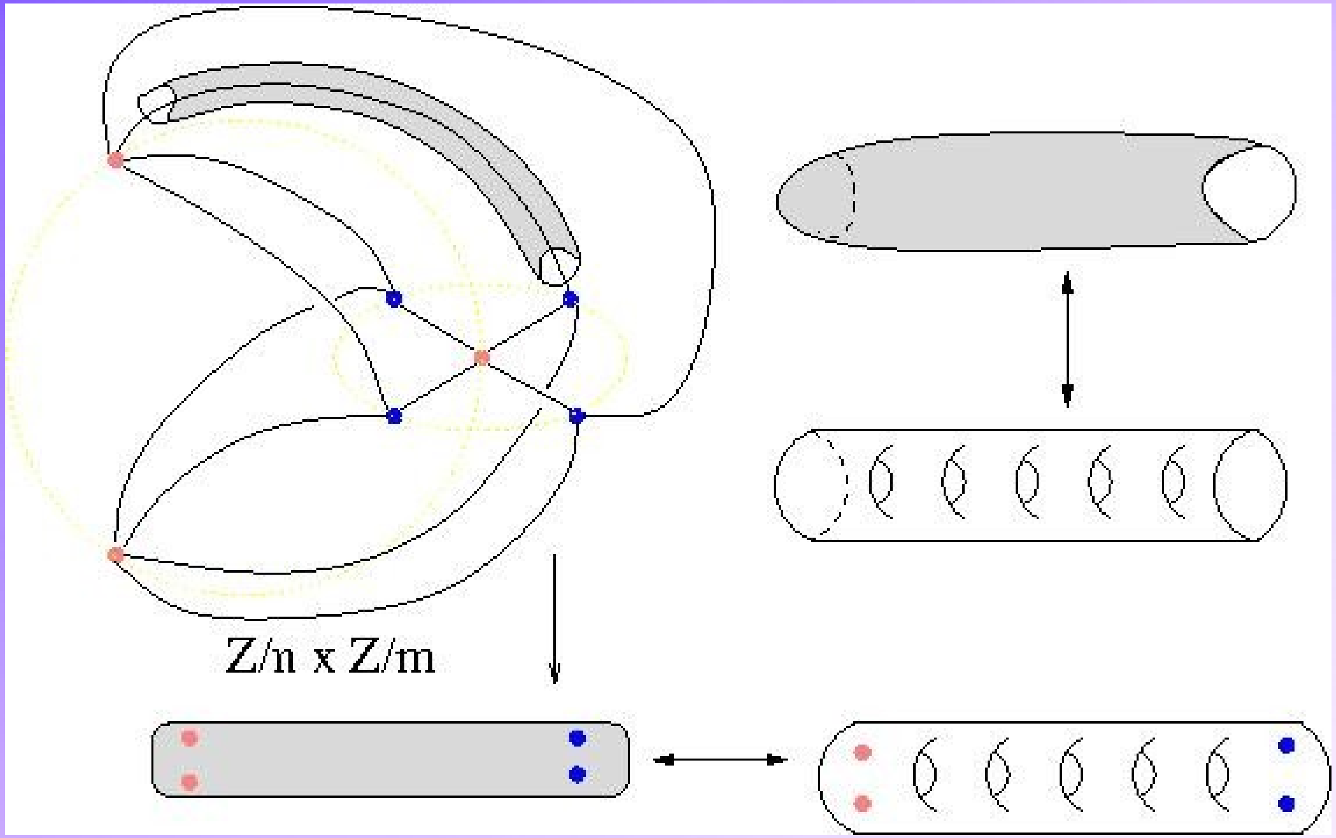
$$\searrow \qquad\qquad \swarrow$$

$$T(2,2,2m,2n)$$

where $g=5mn+(m-1)(n-1)$ with m, $n \geq 2$ ($n \neq m$).

Extension to the branched case and an application

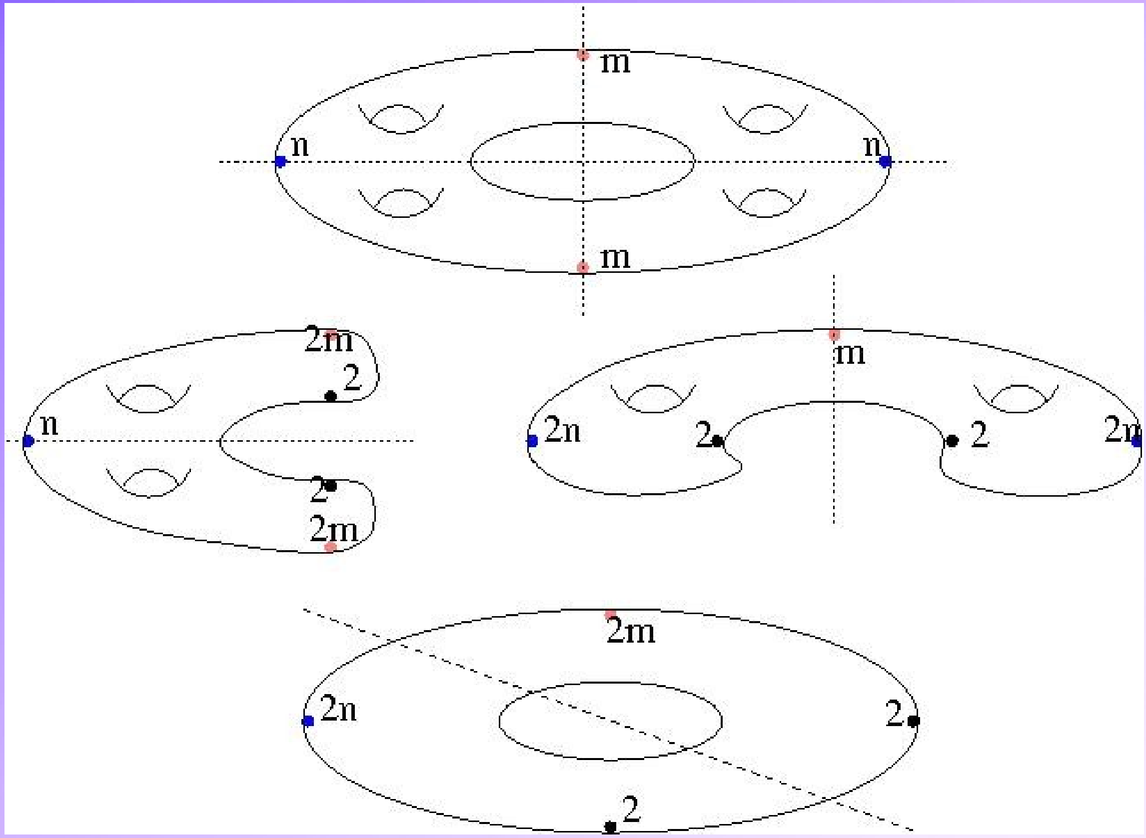
The top part corresponds to



note that the Hopf link admits a **Z**/*n* x **Z**/*m* symmetry for all *n* and *m* (here *n*=4 and *m*=3).

Extension to the branched case and an application

The bottom part corresponds to



note that because of the symmetry, if one forgets the orders, the *m* and *n* cone points can be exchanged

Extension to the branched case and an application

Consider a linear Anosov map. Such a map commutes with the elliptic involution -Id. Up to taking a power, we can assume that the Anosov map has four fixed points that are exchanged in pairs by the elliptic involution.

Identify these fixed points with the cone points of order 2, 2, $2m$, and $2n$ so that the cone points of order 2 belong to the same orbit of the elliptic involution.

Up to choosing a further sufficiently large power, we can lift the Anosov map to pseudo-Anosov maps on all surfaces of the covering system, as in the previous situation.

Extension to the branched case and an application

Remarks

All pseudo-Anosov maps fix point-wise the marked points.

The pseudo-Anosov maps defined on the surfaces of genus 2 are conjugate.

The surface coverings induce coverings of the associated mapping tori.

Remarks

The mapping torus *E,* associated to the surface of genus *g,* is a covering of the mapping torus *B,* associated to the surface of genus 2, branched along a five-component link *L,* transverse to the natural fibration, in two different ways.

The link *L* is the suspension of the five marked points which are fixed by the two conjugate pseudo-Anosov maps. Note that the conjugation identifies the components marked $(2,2,2m,2m,n)$ with those marked $(2,2,2n,2n,m)$ in that order.

Extension to the branched case and an application

In conclusion we have:

Theorem (Los, P., Salgueiro)

There are infinitely many pairs of non-isometric hyperbolic orbifolds with the same volume and topological type.


Thank you
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You can drink