Characteristic classes of flat bundles and invariants of homology spheres

Shigeyuki MORITA

based on jw/w Takuya SAKASAI and Masaaki SUZUKI

October 24, 2017

Contents

A mystery concerning homology 3-spheres (1)

$\mathcal{M}(3) = \{$closed oriented 3-manifold$\}/$ori. pres. diffeo.

$\cup$

$\mathcal{H}(3) = \{$closed oriented homology 3-sphere$\}/$ori. pres. diffeo.

$$\mathbf{\Theta}^3 = \mathcal{H}(3)/\text{smooth H-cobordism} \quad \text{(abelian group)}$$

Theorem (Furuta, Fintushel-Stern)

$\mathbf{\Theta}^3$ *has an infinite rank*

$\Rightarrow \mathbf{\Theta}^3/$torsion $\subset \mathbb{Q}^\infty$ (because $\mathbf{\Theta}^3$ is countable)

$\Rightarrow \mathrm{Hom}(\mathbf{\Theta}^3, \mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}}$ (direct product of countably many $\mathbb{Q}$)

A mystery concerning homology 3-spheres (2)

so there exist (uncountably) many homomorphisms

$$\mathbf{\Theta}^3 \longrightarrow \mathbb{Q}$$

but explicitly known one(s): Frϕyshov and Ozsváth-Szabó

Habiro $\Rightarrow$ no finite type rational invariant (e.g. Casson invariant)

is invariant under H-cobordism

A mystery concerning homology 3-spheres (3)

candidate: Neumann-Siebenmann, Fukumoto-Furuta-Ue, Saveliev

$$\nu := \sum_{i=0}^{7} (-1)^{\frac{i(i+1)}{2}} \operatorname{rank} HF^i \text{ (instanton Floer homology)}$$

recall:

Theorem (Taubes)

$\sum_{i=0}^{7} (-1)^i \operatorname{rank} HF^i = 2\lambda$ *(Casson invariant)*

Theorem (Manolescu)

The Rohlin homomorphism $\Theta^3 \to \mathbb{Z}/2$ *does not split*

A dream (1)

ultimate goal:

want to construct homomorphisms

$$\nu_k : \mathbf{\Theta}^3 \to \mathbb{Q} \quad (k = 1, 2, \ldots)$$

method:

extend the interpretation of the Casson invariant

as a secondary invariant associated with the fact that

the first MMM class vanishes on the Torelli group

in the context of a larger group than the MCG and

new characteristic classes

A dream (2)

$H^2(\mathcal{M}_g;\mathbb{Q}) \ni e_1 \longmapsto 0 \in H^2(\mathcal{I}_g;\mathbb{Q})$ first MMM-class

$\Rightarrow$ secondary invariant $\quad d_1 : \mathcal{K}_g \to \mathbb{Q} \ \sim \lambda : \mathcal{H}(3) \to \mathbb{Z}$

$H^{4k+1}(\mathrm{GL}(N,\mathbb{Z});\mathbb{R}) \ni \beta_{2k+1} \mapsto 0? \in H^{4k+1}(\mathrm{GL}(2k+2,\mathbb{Z});\mathbb{R})$

$\overset{\text{Igusa,Galatius}}{\mapsto} 0 \in H^{4k+1}(\mathrm{Out}\, F_N;\mathbb{R}) \ \Rightarrow$ secondary invariant $\quad \mu_k$

$\mathfrak{t}_{2k+1} \in H^2(\mathfrak{h}_{g,1})_{4k+2} \overset{\text{Kontsevich}}{\cong} H_{4k}(\mathrm{Out}\, F_{2k+2};\mathbb{Q}) \ni \mu_k$

$\longmapsto \tilde{\mathfrak{t}}_{2k+1} \neq ? \in H^2(\mathcal{H}^{\mathrm{smooth}}_{g,1};\mathbb{Q}) \mapsto 0? \in H^2(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q})$

$\Rightarrow$ secondary^2 invariant $\quad \nu_k : \mathbf{\Theta}^3 \to \mathbb{Q}$

Strategy (1)

extending $\mathcal{M}_g \Rightarrow \mathcal{H}_{g,1}$ and $e_1 \Rightarrow \tilde{\mathbf{t}}_{2k+1}$, ultimate goal:

$$H^2(\mathcal{H}_{g,1}^{\text{top}};\mathbb{Q}) \ni \tilde{\mathbf{t}}_{2k+1} \mapsto 0 \in H^2(\mathcal{H}_{g,1}^{\text{smooth}};\mathbb{Q}) \ \Rightarrow \ \nu_k : \mathbf{\Theta}^3 \to \mathbb{Q}$$

Garoufalidis-Levine (based on Goussarov and Habiro)

$$\mathcal{H}_{g,1}^{\text{smooth}} = \{\text{homology cylinder over } \Sigma_{g,1}\}/\text{smooth H-cobordism}$$

$$\mathcal{H}_{0,1}^{\text{smooth}} = \mathbf{\Theta}^3 = \mathcal{H}(3)/\text{smooth H-cobordism} \overset{\text{central}}{\subset} \mathcal{H}_{g,1}^{\text{smooth}}$$

$$\mathbf{\Theta}^3 \to \mathcal{H}_{g,1}^{\text{smooth}} \to \overline{\mathcal{H}}_{g,1} = \mathcal{H}_{g,1}^{\text{smooth}}/\mathbf{\Theta}^3 \quad \text{(central extension)}$$

Strategy (2)

exact sequence:

$$0 \to H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q}) \to H^1(\boldsymbol{\Theta}^3;\mathbb{Q})$$
$$\cong \mathrm{Hom}(\boldsymbol{\Theta}^3,\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}} \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})$$

Problem

How is the huge group $H^1(\boldsymbol{\Theta}^3;\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}}$ divided into

$$\mathrm{Coker}\left(H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})\right) \quad \textit{and}$$
$$\mathrm{Ker}\left(H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})\right) ?$$

Coker is non-trivial $\Leftrightarrow$

$^\exists$ homomorphism $\boldsymbol{\Theta}^3 \to \mathbb{Q}\ (\neq 0)$ which extends to $\mathcal{H}_{g,1}^{\mathrm{smooth}} \to \mathbb{Q}$

Strategy (3)

Mal'cev completion of $\pi_1\Sigma_{g,1}$: $\cdots \to N_d \to \cdots \to N_1 = H_{\mathbb{Q}}$

Theorem (Garoufalidis-Levine)

$\exists\ \tilde{\rho}_\infty : \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d$ (*symplectic auto. groups*)

each factor $\tilde{\rho}_d : \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \mathrm{Aut}_0\, N_d$ *is surjective over* $\mathbb{Z}$

candidates for Ker: constructed a homomorphism

$$\tilde{\rho} : \overline{\mathcal{H}}_{g,1} \to \left(\wedge^3 H_{\mathbb{Q}} \oplus \prod_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$$

and defined

$$(\wedge^2 S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \ni 1 \mapsto \tilde{\mathbf{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

Strategy (4)

replacing $\overline{\mathcal{H}}_{g,1}$ with more geometric object

(2008, after a comment by Orr):

$\mathcal{H}_{g,1}^{\rm top} = \{$homology cylinder over $\Sigma_{g,1}\}/$topological H-cobordism

Theorem (Freedman)

Any homology 3-sphere bounds a contractible topological 4-mfd

It follows that $\mathcal{H}_{g,1}^{\rm smooth} \rightarrow \mathcal{H}_{g,1}^{\rm top}$ factors through $\overline{\mathcal{H}}_{g,1}$

$$\mathbf{\Theta}^3 \rightarrow \mathcal{H}_{g,1}^{\rm smooth} \rightarrow \overline{\mathcal{H}}_{g,1} \rightarrow \mathcal{H}_{g,1}^{\rm top}$$

and the homomorphisms $\tilde{\rho}_\infty, \tilde{\rho}$ are actually defined on $\mathcal{H}_{g,1}^{\rm top}$

$$\Rightarrow \quad \tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{top};\mathbb{Q})$$

Strategy (5)

Theorem (Sakasai-Suzuki-M.)

$$\exists\ \tilde{\rho}_\infty^* : H_c^*(\hat{\mathfrak{h}}_{\infty,1}^+)^{\mathrm{Sp}} \otimes H^*(\mathrm{Sp}(2\infty,\mathbb{Z})) \to H^*(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q})$$

$$\Rightarrow\ H_c^2(\hat{\mathfrak{h}}_{\infty,1}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q}) \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

$H_c^2(\hat{\mathfrak{h}}_{\infty,1}) \ni \mathfrak{t}_{2k+1}$ (Lie algebra version) $\mapsto \tilde{\mathfrak{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$

$$H_c^2(\hat{\mathfrak{h}}_{\infty,1})_{4k+2} \overset{\text{Kontsevich}}{\cong} H_{4k}(\mathrm{Out}\, F_{2k+2};\mathbb{Q})$$

$\ni \mathfrak{t}_{2k+1}$ $\qquad\qquad$ $\ni \mu_k$ Morita classes

only $k = 1, 2, 3$ are known to be non-trivial (using computer) as the first step, want to prove the non-trivialities of

$$\mathbf{t}_{2k+1} \cong \mu_k \quad (k = 4, 5, \ldots)$$

computer computation is at present hopeless

because it requires huge memories and time

trying to prove the non-triviality by considering

possible geometric meaning of μ_k and then

utilize it

Characteristic classes of flat vector bundles

$\beta_{2k+1} \in H^{4k+1}(\mathrm{GL}(n,\mathbb{R})^\delta;\mathbb{R})$ (Borel regulator class)

Theorem (Borel)

$$\lim_{n\to\infty} H^*(\mathrm{GL}(n,\mathbb{Z});\mathbb{R}) \cong \wedge_{\mathbb{R}}(\beta_3,\beta_5,\ldots)$$

what is the stable range ? of the Borel classes

We can also consider the Euler class for comparison

$\chi \in H^{2n}(\mathrm{SL}(2n,\mathbb{R})^\delta;\mathbb{Q})$ (unstable class)

what is the critical monodromy ? of the Euler class

Critical monodromy and dimension for non-triviality (1)

$$H^*(\mathfrak{gl}(k,\mathbb{R}),\mathrm{O}(k)) \to H^*(\mathrm{GL}(k,\mathbb{R})^\delta;\mathbb{R})$$

$$\begin{array}{rll}
H^*(\mathfrak{gl}(2k+1,\mathbb{R}),\mathrm{O}(2k+1)) \cong H^*(S^1 \times & S^5 \times \cdots \times S^{4k+1};\mathbb{Q}) & \\
& \beta_3 \quad \cdots \quad \beta_{2k+1} & \\
H^*(\mathfrak{gl}(2k+2,\mathbb{R}),\mathrm{O}(2k+2)) \cong H^*(S^1 \times & S^5 \times \cdots \times S^{4k+1} \times S^{2k+2};\mathbb{Q}) & \\
& \beta_3 \quad \cdots \quad \beta_{2k+1} \quad\quad \chi & \\
H^*(\mathfrak{gl}(2k+3,\mathbb{R}),\mathrm{O}(2k+3)) \cong H^*(S^1 \times & S^5 \times \cdots \times S^{4k+1} \times S^{4k+5};\mathbb{Q}) & \\
& \beta_3 \quad \cdots \quad \beta_{2k+1} \quad\quad \beta_{2k+3} &
\end{array}$$

Critical monodromy and dimension for non-triviality (2)

$$H^{4k+1}(\mathrm{GL}(2k,\mathbb{Z})) \leftarrow H^{4k+1}(\mathrm{GL}(2k+1,\mathbb{Z})) \leftarrow$$

0 (form level) $\quad\quad \beta_{2k+1} = 0$

(Lee), Bismut-Lott

$$\leftarrow H^{4k+1}(\mathrm{GL}(2k+2,\mathbb{Z})) \leftarrow H^{4k+1}(\mathrm{GL}(2k+3,\mathbb{Z}))$$

$k = 1;$ $\quad \beta_3 = 0$ (Lee-Szczarba) $\quad \beta_3 \neq 0$ (EGS?)

$k = 2;$ $\quad \beta_5 = 0$, Elbaz-Gangl-Soulé $\quad \beta_5 \neq 0$ (EGS?)

$k \geq 3;$ $\quad \beta_{2k+1} = 0$? (unknown) $\quad \beta_{2k+1} \neq 0$ (Lee)

Critical monodromy and dimension for non-triviality (3)

Sullivan:

$$\chi = 0 \in H^{2n}(\mathrm{SL}(2n,\mathbb{Z});\mathbb{Q})$$

Milnor:

$$\chi \neq 0 \in H^{2}(\mathrm{SL}(2,\mathbb{Z}[1/2]);\mathbb{Q})$$

$\Rightarrow$

$$\chi \neq 0 \in H^{2n}(\mathrm{SL}(2n,\mathbb{Z}[1/2]);\mathbb{Q}) \quad \text{(for any } n\text{)}$$

because

$$H^{2}(\mathrm{SL}(2,\mathbb{Z}[1/2]);\mathbb{Q}) \otimes \cdots \otimes H^{2}(\mathrm{SL}(2,\mathbb{Z}[1/2]);\mathbb{Q}) \ni \chi^{\otimes n}$$

$$\mapsto \chi \in H^{2n}(\mathrm{SL}(2n,\mathbb{Z}[1/2]);\mathbb{Q})$$

Critical monodromy and dimension for non-triviality (4)

Theorem (Moss)

Let R be a ring in which 2 and 3 are invertible, then

$$H_i(\mathrm{SL}(2,\mathbb{Z}[1/2]);R) = \begin{cases} R & (i=0,2) \\ 0 & \textit{otherwise} \end{cases}$$

$$H_i(\mathrm{SL}(3,\mathbb{Z}[1/2 \textit{ or } 1/3]);R) = \begin{cases} R & (i=0,5) \\ 0 & \textit{otherwise} \end{cases}$$

thus $\mathbb{Z}$ and $\mathbb{Z}[1/2]$ coefficients cases are completely different!

Conjectural meaning of Morita classes (1)

geometric meaning of $\mu_k \in H_{4k}(\mathrm{Aut}\, F_{2k+2};\mathbb{Q})$

Conjecture

$\mu_k \in H_{4k}(\mathrm{Aut}\, F_{2k+2};\mathbb{Q})$ *can be detected by a secondary class associated with the two different reasons for the Borel class* $\beta_{2k+1} \in H^{4k+1}(\mathrm{GL}(N,\mathbb{Z});\mathbb{R})$ *to vanish in* $H^{4k+1}(\mathrm{Aut}\, F_N;\mathbb{R})$ *and* $H^{4k+1}(\mathrm{GL}(2k+2,\mathbb{Z});\mathbb{R})$*, namely*

$$\langle (p_0)_* j^* z_{4k} - x_{4k}, \mu_k \rangle \neq 0 \;?$$

$$\begin{array}{ccc} Z^{4k}(\mathrm{Aut}\, F_{2k+2};\mathbb{R}) & \overset{j^*}{\longleftarrow} & \delta C^{4k}(\mathrm{Aut}\, F_{2k+3};\mathbb{R}) \\ p_0^* \big\uparrow \text{“partial map”} & & p^* \big\uparrow \text{“partial map”} \\ \delta C^{4k}(\mathrm{GL}(2k+2,\mathbb{Z});\mathbb{R}) & \overset{\bar{j}^*}{\longleftarrow} & Z^{4k+1}(\mathrm{GL}(2k+3,\mathbb{Z});\mathbb{R}) \end{array}$$

$$\begin{array}{ccc} \delta(j^* z_{4k} - p_0^* x_{4k}) = 0 & \overset{j^*}{\longleftarrow} & p^* b_{2k+1} = \delta z_{4k} \\ p_0^* \big\uparrow & & p^* \big\uparrow \\ \bar{j}^* b_{2k+1} \overset{?}{=} \delta x_{4k} & \overset{\bar{j}^*}{\longleftarrow} & b_{2k+1} \end{array}$$

$$\langle (p_0)_* j^* z_{4k} - x_{4k}, \mu_k \rangle \neq 0 \ ?$$

Conjectural meaning of Morita classes (3)

supporting side-evidence

Theorem (Sakasai-Suzuki-M.)

The even MMM *class* $e_{2k} \in H^{4k}(\mathcal{M}_{g,*})$ *can be interpreted as a secondary class associated with the two reasons for the Borel class* $\beta_{2k+1} \in H^{4k+1}(\mathrm{GL}(2g,\mathbb{Z});\mathbb{R})$ *to vanish in* $H^{4k+1}(\mathrm{Aut}\,F_{2g};\mathbb{R})$ *and* $H^{4k+1}(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{R})$, *more precisely*

$$[j^* z_{4k} - p_0^* y_{4k}] = \frac{(-1)^k}{2(2k)!}\,\zeta(2k+1)\,e_{2k} \in H^{4k}(\mathcal{M}_{g,1};\mathbb{R})$$

proof depends on Igusa's higher Franz-Reidemeister torsion

Conjectural meaning of Morita classes (4)

$$[j^* z_{4k} - p_0^* y_{4k}] = \frac{(-1)^k}{2(2k)!}\, \zeta(2k+1)\, e_{2k} \in H^{4k}(\mathcal{M}_{g,1};\mathbb{Q})$$

$$\begin{array}{ccc}
Z^{4k}(\mathcal{M}_{g,*};\mathbb{R}) & \xleftarrow{\ j^*\ } & \delta C^{4k}(\mathrm{Aut}\, F_{2g};\mathbb{R}) \\
p_0^* \Big\uparrow \text{“partial map”} & & p^* \Big\uparrow \text{“partial map”} \\
\delta C^{4k}(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{R}) & \xleftarrow{\ \bar{j}\ } & Z^{4k+1}(\mathrm{GL}(2g,\mathbb{Z});\mathbb{R})
\end{array}$$

$$\begin{array}{ccc}
\delta(j^* z_{4k} - p_0^* y_{4k}) = 0 & \longleftarrow & p^* b_{2k+1} = \delta z_{4k} \\
\Big\uparrow & & \Big\uparrow \\
\bar{j}^* b_{2k+1} = \delta y_{4k} & \longleftarrow & b_{2k+1}
\end{array}$$

Conjectural meaning of Morita classes (5)

conjectural geometric meaning of μ_k (dual version), based on:

Theorem (Conant-Vogtmann)

$H_{4k}(\mathrm{Aut}\, F_{2k+2}; \mathbb{Q}) \ni \tilde{\mu}_k$ (*lift of* μ_k) $\mapsto 0 \in H_{4k}(\mathrm{Aut}\, F_{2k+3}; \mathbb{Q})$

Theorem (Conant-Hatcher-Kassabov-Vogtmann)

$\tilde{\mu}_k \in H_{4k}(\mathrm{Aut}\, F_{2k+2}; \mathbb{Q})$ *is supported on a certain free abelian subgroup* $\mathbb{Z}^{4k} \subset \mathrm{Aut}\, F_{2k+2}$

Conjectural meaning of Morita classes (6)

Conjecture

$$p_*(\mu_k) = 0 \in H_{4k}(\mathrm{GL}(2k+2,\mathbb{Z});\mathbb{Q}) \quad \textit{and}$$

$$\langle u_{2k+1}, \beta_{2k+1}\rangle \neq 0 \quad (\Rightarrow \mu_k \neq 0)$$

$$\begin{array}{ccc}
Z_{4k}(\mathrm{Aut}\, F_{2k+2};\mathbb{Q}) & \xrightarrow{\tilde{\mu}_k \mapsto \partial u^f_{2k+1}} & \partial C_{4k+1}(\mathrm{Aut}\, F_{2k+3};\mathbb{Q}) \\
{\scriptstyle \tilde{\mu}_k \mapsto \partial u^b_{2k+1}} \Big\downarrow {\scriptstyle p_*} & & {\scriptstyle p_*} \Big\downarrow \\
\partial C_{4k+1}(\mathrm{GL}(2k+2,\mathbb{Z});\mathbb{Q}) & \xrightarrow{i_*} & Z_{4k+1}(\mathrm{GL}(2k+3,\mathbb{Z});\mathbb{Q})
\end{array}$$

$$u_{2k+1} = [p_* u^f_{2k+1} - i_* u^b_{2k+1}] \in H_{4k+1}(\mathrm{GL}(2k+3,\mathbb{Z});\mathbb{Q})$$

Bird's-eye view

$2k+1$	1	3	5	$\cdots$
weight $(4k+2)$	2	6	10	$\cdots$
generators of $\mathfrak{h}_{g,1}, \sqrt{\text{Galois}}$	$\Lambda^3 H/H$	$S^3 H$	$S^5 H$	$\cdots$
period	$\zeta(1)$	$\zeta(3)$	$\zeta(5)$	$\cdots$
Soulé (Galois image)		σ_3	σ_5	$\cdots$
$H^2(\mathcal{H}_{\infty,1})_{4k+2}$	e_1	$\tilde{\mathbf{t}}_3$	$\tilde{\mathbf{t}}_5$	$\cdots$
$H_{4k}(\mathrm{Out}\, F_{2k+2})$		μ_1	μ_2	$\cdots$
Borel class		β_3	β_5	$\cdots$
3-dim. invariant	λ	ν_3	ν_5	$\cdots$

$\mathfrak{h}_{g,1}(2k+1) \supset S^{2k+1} H_{\mathbb{Q}}$ (trace component)

$(\wedge^2 S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \xrightarrow{[\ ,\ ]} \sigma^{\mathrm{top}}_{2k+1} \subset \mathfrak{h}_{g,1}(4k+2)$ Galois image ?

$\mapsto \mathbf{t}_{2k+1} \in H^2(\mathfrak{h}_{g,1})_{4k+2} \cong \mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2})$

$\mapsto \tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}^{\mathrm{top}}_{g,1})_{4k+2} \to H^2(\mathcal{H}^{\mathrm{smooth}}_{g,1})_{4k+2}$