On a Nielsen-Thurston classification theory on cluster modular groups

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§0. Background

Cluster ensembles (Fock-Goncharov '03~)

Q : a quiver

$\rightsquigarrow$ (1) $\underset{\text{presymplectic}}{\mathcal{A}_{|Q|}(\mathbb{R}_{>0})}$, $\underset{\text{Poisson}}{\mathcal{X}_{|Q|}(\mathbb{R}_{>0})}$: a pair of mfd's

($|Q|$: "mutation class")

related by a map $p : \mathcal{A}_{|Q|}(\mathbb{R}_{>0}) \longrightarrow \mathcal{X}_{|Q|}(\mathbb{R}_{>0})$

"cluster ensemble"

(2) $\Gamma_{|Q|}$: a discrete group which acts on $\mathcal{A} \xrightarrow{p} \mathcal{X}$

"cluster modular group"

Typically we also get a symplectic manifold :

$$\begin{tikzcd} \mathcal{A}_{|Q|}(\mathbb{R}_{>0}) \arrow[d] \arrow[dr, "p"] & \\ p(\mathcal{A}_{|Q|}(\mathbb{R}_{>0})) \arrow[r, hook] & \mathcal{X}_{|Q|}(\mathbb{R}_{>0}) \end{tikzcd}$$

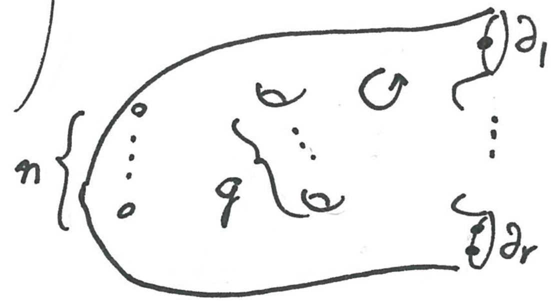
$$\begin{gathered} p(\mathcal{A}_{|Q|}(\mathbb{R}_{>0})) \\ \parallel \\ \mathcal{U}_{|Q|}(\mathbb{R}_{>0}) \ : \ \text{a symplectic manifold.} \end{gathered}$$

* Today we mainly deal w/ the $\mathcal{X}$-side.

The constructions for the $\mathcal{A}$-side are similar.

③ Relation with the decorated Teichmüller theory [Penner, Fock, ⋯]

$\Sigma = \Sigma^{n}_{g\vec{\delta}}$ $\left(\begin{array}{l} \vec{\delta} = (\delta_1, \cdots, \delta_r), \quad \delta_j := \#\text{ marked points on } \partial_j \\ 2-2g-\sum_{j=1}^{r}\delta_j - n < 0, \quad n + \sum_{j=1}^{r}\delta_j \geq 1 \end{array} \right)$



$\rightsquigarrow$ $^{\exists}Q$ s.t. :

$\mathcal{A}_{|Q|}(\mathbb{R}_{>0}) \cong \widetilde{T}(\Sigma)$: the decorated Teichmüller space via Penner's λ-length coordinates

$$\boxed{\begin{tikzcd} \widetilde{T}(\Sigma) \arrow[d] \arrow[dr, "p"] & \\ T(\Sigma) \arrow[r, hook] & \widehat{T}(\Sigma) \end{tikzcd}}$$

$\mathcal{X}_{|Q|}(\mathbb{R}_{>0}) \cong \widehat{T}(\Sigma)$: the enhanced Teichmüller space via Fock-Thurston's cross ratio coordinates

$\mathcal{U}_{|Q|}(\mathbb{R}_{>0}) \cong T(\Sigma)$: the Teichmüller space

$\Gamma_{|Q|} \cong MC^{\bowtie}(\Sigma)$: the tagged mapping class group $\cong MC(\Sigma)$ if $\left(\begin{array}{c} n=0 \\ \text{or} \\ n=1, \ \partial\Sigma = \phi \end{array} \right)$

Main results

* We define 3 types of elements of $\Gamma_{|Q|}$

* We characterize them in terms of the action

$$\Gamma_{|Q|} \curvearrowright \overline{\mathcal{X}}_{|Q|} \ (\approx \text{a closed disk})$$

Fock-Goncharov compactification

- Detect ∞-order elements using an algebraic equation.
- Relate the "cluster complex" w/ the FG boundary.

<u>Related problem</u> (Papadopoulos-Penner '93)

Characterize the NT classification of mapping classes in terms of (tropical) λ-length coordinates.

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§1. Cluster ensembles.

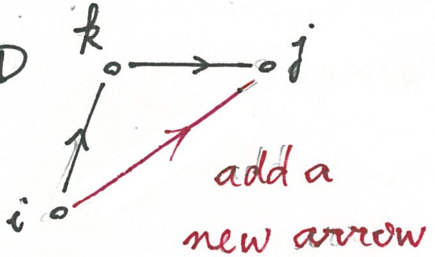
Let $Q = (I, \varepsilon)$ be a quiver w/o loops, 2-cycles, where

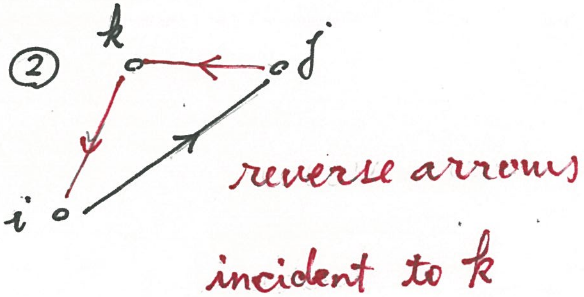
$$\begin{cases} I : \text{a finite set} \\ \varepsilon : I \times I \longrightarrow \mathbb{Z} \quad \text{skew-symmetric} \end{cases} \qquad \left(\begin{array}{l} I = \text{the vertex set} \\ \varepsilon_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\} \end{array} \right)$$

In some cases we also take a subset $I_0 \subset I$ "frozen vertices" on which mutations are forbidden.

<u>Def</u> (mutations)

Q : a quiver, $k \in I \setminus I_0 \rightsquigarrow \mu_k(Q) = (I, \varepsilon')$: a new quiver is given by

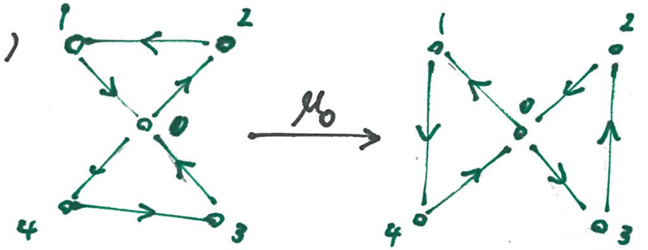
① 

② 

③ Delete 2-cycles's

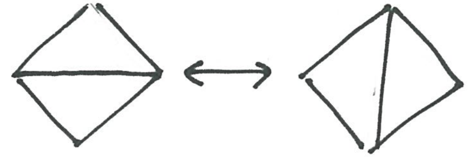
Examples

(1)



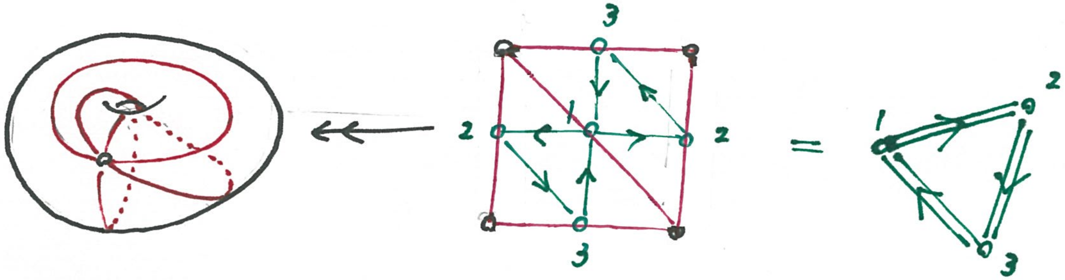
★ mutation

= flip :

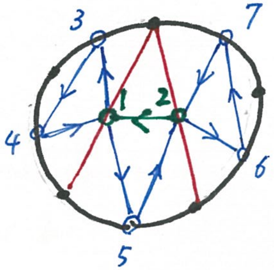


(2) For $S = S^n_{g,\vec{\delta}}$, take an ideal triangulation Δ.

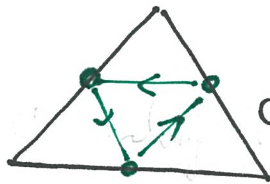
e.g. $S = S^1_{1,\vec{0}}$



e.g. $S = S^0_{0,(5)}$



Then the quiver $Q = Q_\Delta$ is obtained by glueing



$\circlearrowleft$ along Δ.

$I_0 = \{$boundary segments$\}$ (frozen)

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Q : a quiver $\rightsquigarrow \mathcal{X}_Q := \mathbb{R}_{>0}^{I \setminus I_0}$ $\qquad (X_i)_{i \in I \setminus I_0}$: coordinates on $\mathcal{X}_Q$

<u>Def</u> (cluster $\mathcal{X}$-transformations)

$\mu_k : Q \longrightarrow Q'$

$$\rightsquigarrow \mu_k^{x} : \mathcal{X}_Q \xrightarrow{\sim} \mathcal{X}_{Q'} \quad : \text{given by } (\mu_k^{x})^* X_i' = \begin{cases} X_k^{-1} & (i=k) \\ X_i\left(1+X_k^{\operatorname{sgn}\varepsilon_{ki}}\right)^{\varepsilon_{ki}} & (i \neq k) \end{cases}$$

<u>Example</u> (type A_2)

$$\begin{array}{ccccccccccc} \overset{0}{\circ} & \longrightarrow & \overset{1}{\circ} & \overset{\mu_0}{\longleftrightarrow} & \overset{0}{\circ} & \longleftarrow & \overset{1}{\circ} & \overset{\mu_1}{\longleftrightarrow} & \overset{0}{\circ} & \longrightarrow & \overset{1}{\circ} & \cdots\cdots \\ X_0 & & X_1 & & X_0^{-1} & & X_1(1+X_0) & & X_0^{-1}[1+X_1(1+X_0)] & & X_1^{-1}(1+X_0)^{-1} & \end{array}$$

$|Q| := \{$ quivers obtained from Q by a sequence of mutations $\}$

Def (cluster modular groups)

$$\Gamma_{|Q|} := \left\{ \begin{array}{l} \text{sequence of mutations} \\ \quad \& \text{ relabellings of vertices} \end{array} \;\middle|\; \text{preserves the quiver} \right\} \Big/ \sim$$

where $\phi_1 \sim \phi_2 \;:\Leftrightarrow\; \phi_1^x = \phi_2^x : \mathcal{X}_Q \longrightarrow \mathcal{X}_Q$

$$\left(\phi = \sigma \cdot \mu_{i_k} \cdots \mu_{i_1} \;\Longrightarrow\; \phi^x := \underset{=}{\sigma^x} \cdot \mu_{i_k}^x \cdots \mu_{i_1}^x \right)$$

permutation of coordinates

Def (cluster $\mathcal{X}$-space)

$\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ is a manifold s.t.

- $\forall Q' \in |Q|$, we have a diffeom. $\mathcal{X}_{Q'} : \mathcal{X}_{|Q|}(\mathbb{R}_{>0}) \xrightarrow{\sim} \mathcal{X}_{Q'}$
- if $Q'' = \mu_k Q'$, we have

$$\begin{tikzcd}
& \mathcal{X}_{Q'} \arrow[d, "\mu_k^x"] \\
\mathcal{X}_{|Q|}(\mathbb{R}_{>0}) \arrow[ur, "\mathcal{X}_{Q'}"] \arrow[r, "\mathcal{X}_{Q''}"'] & \mathcal{X}_{Q''}
\end{tikzcd}$$

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$*\ \Gamma_{|Q|} \overset{C^\infty}{\curvearrowright} \mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ naturally.

<u>Remark</u> $\{X_i, X_j\}_{\mathcal{X}} := \varepsilon_{ij} X_i X_j$ defines a $\Gamma_{|Q|}$-inv. Poisson structure on $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$.

<u>Examples</u>

(1) $Q_{A_2} := \overset{0}{\circ} \longrightarrow \overset{1}{\circ}$, $\phi := (0\ 1)\mu_0 \in \Gamma_{A_2}$ <u>Fact</u> $\phi^5 = 1$ "pentagon identity"

The action on $\mathcal{X}_{A_2}(\mathbb{R}_{>0})$ is given by $\phi^x \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} X_1(1+X_0) \\ X_0^{-1} \end{pmatrix}$

(2) $Q_{I_2(k)} := \overset{0}{\circ} \xrightarrow{k} \overset{1}{\circ}$ $(k \geq 2)$, $\phi := (0\ 1)\mu_0 \in \Gamma_{I_2(k)}$ <u>Fact</u> ϕ has ∞-order.

The action on $\mathcal{X}_{I_2(k)}(\mathbb{R}_{>0})$ is given by $\phi^x \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} X_1(1+X_0)^k \\ X_0^{-1} \end{pmatrix}$

<u>Observation</u> The fixed point equation $\phi^x \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}$ has a positive solution in (1), but does NOT in (2).

Is it related to the <u>finite-orderness</u> of ϕ ?

<u>Def</u> (Fock-Goncharov compactification)

$\mathcal{X}_{|Q|}(\mathbb{R}^t)$, called the tropical space, is a PL mfd s.t.

- $\forall Q' \in |Q|$, we have a homeom. $x_{Q'} : \mathcal{X}_{|Q|}(\mathbb{R}^t) \xrightarrow{\sim} \mathbb{R}^{I \setminus I_0}$
- if $Q'' = \mu_k Q'$, we have

$$\begin{tikzcd} & \mathbb{R}^{I\setminus I_0} \arrow[d, "\mathrm{Trop}\,\mu_k^x"] \\ \mathcal{X}_{|Q|}(\mathbb{R}^t) \arrow[ur, "x_{Q'}"] \arrow[r, "x_{Q''}"'] & \mathbb{R}^{I\setminus I_0} \end{tikzcd}$$

where $(\mathrm{Trop}\,\mu_k^x)^* x_i' := \begin{cases} -x_k & (i=k) \\ x_i + \varepsilon_{ki} \max\{0, (\mathrm{sgn}\,\varepsilon_{ki}) x_i\} & (i \neq k) \end{cases}$

$\rightsquigarrow \overline{\mathcal{X}}_{|Q|} := \mathcal{X}_{|Q|}(\mathbb{R}_{>0}) \sqcup \underbrace{P\mathcal{X}_{|Q|}(\mathbb{R}^t)}_{\substack{\parallel \\ (\mathcal{X}_{|Q|}(\mathbb{R}^t) \setminus \{0\}) / \mathbb{R}_{>0}}}$: the Fock-Goncharov compactification.

Here, the topology is that of the log-spherical compactification.

<u>Lemma</u> · $\overline{\mathcal{X}}_{|\mathfrak{Q}|} \approx D^{I \setminus I_0}$: a closed disk

· $\Gamma_{|\mathfrak{Q}|} \overset{C^0}{\curvearrowright} \overline{\mathcal{X}}_{|\mathfrak{Q}|}$

sketch of proof)

The "log-cluster transformation" $(x_i := \log X_i)$

$$x_i' = \begin{cases} -x_k & (i=k) \\ \log(1+e^{(\mathrm{sgn}\,\varepsilon_{ki})x_i})^{\varepsilon_{ki}} & (i \neq k) \end{cases}$$

asymptotically coincides w/ $\mathrm{Trop}\,\mu_k^x$ as $|x_i| \gg 1$. ▨

<u>Cor</u> $\phi \in \Gamma_{|\mathfrak{Q}|}$: finite order $\Rightarrow \exists x \in \mathcal{X}_{|\mathfrak{Q}|}(\mathbb{R}_{>0})$ s.t. $\phi(x) = x$.

sketch of proof)

* By Brouwer's fixed point theorem, each $\phi \in \Gamma_{|Q|}$ has a fixed point in $\overline{X}_{|Q|} \approx D^{I \setminus I_0}$.

* Suppose $\phi \in \Gamma_{|Q|}$ has fin. order & has no fixed point in $X_{|Q|}(\mathbb{R}_{>0}) = \operatorname{int} D^{I \setminus I_0}$.

$$\begin{tikzcd}
D^n \arrow[r, "\phi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & D^n \arrow[d, "\pi"] \\
S^n = D^n/\partial D^n \arrow[r, "\exists \tilde{\Phi}"'] & D^n/\partial D^n = S^n
\end{tikzcd}$$

$\tilde{\Phi}$ has only one fixed point $\pi(\partial D^n) \in S^n$.

$(n := |I \setminus I_0|)$

* Then it contradicts to Brown's fixed point formula:

$$\underset{\underset{\chi(pt)=1}{\parallel}}{\chi(\operatorname{Fix}(\tilde{\Phi}))} = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(\phi_* : H_i(S^n) \longrightarrow H_i(S^n))$$ for fin. order homeom's

even

$\blacksquare$

⑪ Constructions for the $\mathcal{A}$-side.

$$Q \rightsquigarrow \mathcal{A}_Q := \mathbb{R}_{>0}^{I} \qquad \left(\underline{\text{cf.}}\ \mathcal{X}_Q = \mathbb{R}_{>0}^{I\setminus I_0}\right)$$

$$\mu_k \rightsquigarrow \mu_k^a : \mathcal{A}_Q \xrightarrow{\sim} \mathcal{A}_{Q'}, \qquad (\mu_k^a)^* A_i' := \begin{cases} A_k^{-1}\left(\prod_{j:\,\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j:\,\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}\right) & (i=k) \\ A_i & (i\neq k) \end{cases}$$

- We define $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ by glueing $\mathcal{A}_Q$'s using μ_k^a's.

- Also we have $\overline{\mathcal{A}}_{|Q|} = \mathcal{A}_{|Q|}(\mathbb{R}_{>0}) \sqcup P\mathcal{A}_{|Q|}(\mathbb{R}^{t}) \approx D^I$

$C^0 \circlearrowleft$

$\Gamma_{|Q|}$

★ A_i ($i \in I_0$) are unchanged. "coefficients"

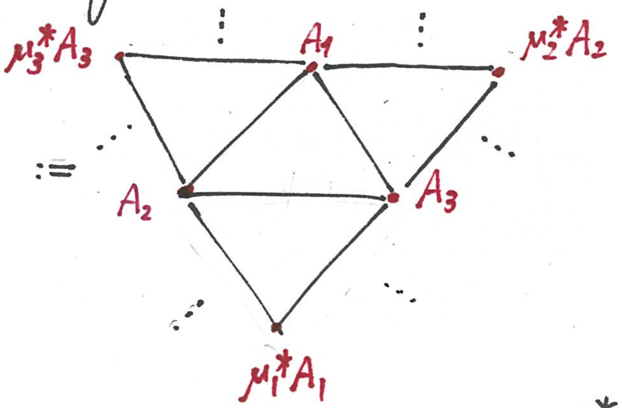
<u>Example</u> (type A_2)

$$\begin{array}{ccccc} \overset{0}{\circ}\longrightarrow\overset{1}{\circ} & \overset{\mu_0}{\longleftrightarrow} & \overset{0}{\circ}\longleftarrow\overset{1}{\circ} & \overset{\mu_1}{\longleftrightarrow} & \overset{0}{\circ}\longrightarrow\overset{1}{\circ} \\ A_0 \qquad A_1 & & \dfrac{A_1+1}{A_0} \qquad A_1 & & \dfrac{A_1+1}{A_0} \qquad \dfrac{\frac{A_1+1}{A_0}+1}{A_1} = \dfrac{A_0+A_1+1}{A_0A_1} \end{array}$$

§2. Cluster complexes & Nielsen-Thurston types

For a quiver Q, define the cluster complex as follows:

$\tilde{\mathcal{C}}_{|Q|} :=$



* Glue ∞'ly many $(|I\setminus I_0|-1)$-dim. standard simplices so that the dual graph $\tilde{\mathcal{C}}^\vee$ is a tree.

* Assign $\mathcal{A}$-variables so that each reflection corresponds to a mutation.

<u>Def</u>

$\mathcal{C}_{|Q|} := \tilde{\mathcal{C}}_{|Q|} / (\text{cells w/ the same variables})$: cluster complex

* $\Gamma_{|Q|} \curvearrowright \mathcal{C}_{|Q|}$ simplicially

(mutation $\longmapsto$ reflection, relabelling $\longmapsto$ permutation)

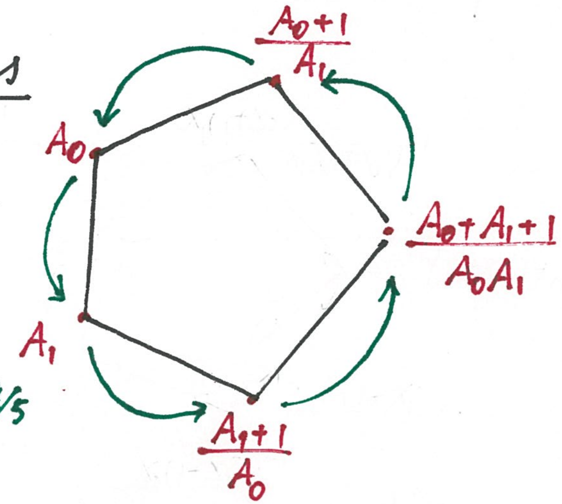
Def (Nielsen-Thurston types)

$\phi \in \Gamma_{|Q|}$ is called cluster-reducible $:\Longleftrightarrow \exists c \in \mathcal{C}_{|Q|}\ \ \phi(c) = c$

cluster-pA $:\Longleftrightarrow \forall n \in \mathbb{Z}\ \ \phi^n$ is cluster-irreducible.

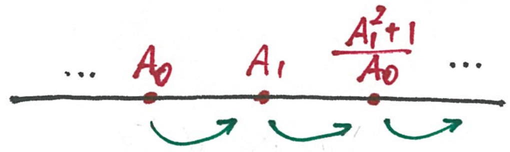
Examples

(1) $\mathcal{C}_{A_2} =$



$\langle \phi \rangle \cong \mathbb{Z}/5$
$\quad \| $
$(0\,1)\mu_0$

(2) $\mathcal{C}_{I_2(k)} =$



$\langle \phi \rangle \cong \mathbb{Z}$
$\quad \|$
$(0\,1)\mu_0$

ϕ is <u>cluster-pA</u>.

(3)
$$\begin{tikzcd}
(S,\Delta) \arrow[r, rightsquigarrow] & \mathcal{C}_{|Q_\Delta|} \cong \mathrm{Arc}^{\bowtie}(S) \arrow[d] & \curvearrowleft \ \mathrm{MC}^{\bowtie}(S) \\
 & \mathrm{Arc}(S) & \curvearrowleft \ \mathrm{MC}(S) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$\boxed{\begin{array}{l} \text{For } \phi \in \mathrm{MC}(S), \\ \text{cluster-red.} \underset{\not\Leftarrow}{\Longrightarrow} \text{reducible} \end{array}}$$

Theorem (I.)

Let Q be a quiver, $\phi \in \Gamma_{|Q|}$ an element.

Then we have the followings :

(1) ϕ : periodic $\Leftrightarrow$ ϕ has a fixed point in $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$.

(2) ϕ : cluster-red. $\Leftrightarrow$ —— 〃 —— in $P\mathcal{X}_{|Q|}(\mathbb{R}^t)_+ \subset P\mathcal{X}_{|Q|}(\mathbb{R}^t)$.

(3) ϕ : cluster-pA $\Leftrightarrow$ $\forall n \in \mathbb{Z}$ ϕ^n —— 〃 —— in $P\mathcal{X}_{|Q|}(\mathbb{R}^t) \setminus P\mathcal{X}_{|Q|}(\mathbb{R}^t)_+$

Here $P\mathcal{X}_{|Q|}(\mathbb{R}^t)_+ := \{ [l] \mid \exists Q' \in |Q| \ \ x_Q(l) \geq 0 \} \subset P\mathcal{X}_{|Q|}(\mathbb{R}^t)$

Moreover if Q is of "Teichmüller type",

then the converse implications hold.

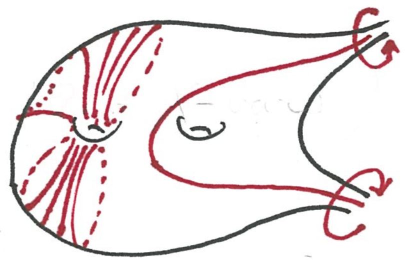
sketch of proof)

Fact In surface cases, $\mathcal{X}_{|Q|}(\mathbb{R}^t) \cong \mathcal{ML}(S)$: the space of measured laminations

(w/ possibly non-compact supports)

For a general quiver, we can define

a $\Gamma_{|Q|}$-equiv. map $\Psi : \mathcal{C}_{|Q|} \longrightarrow P\mathcal{X}_{|Q|}(\mathbb{R}^t)$



which generalizes the embedding $\mathrm{Arc}^{\bowtie}(S) \hookrightarrow \mathbb{P}\mathcal{ML}(S)$.

* We then have $\mathrm{im}\,\Psi = P\mathcal{X}_{|Q|}(\mathbb{R}^t)_+ \quad \Rightarrow (2)_{/\!/} \Rightarrow (3)_{/\!/}$

* "Teichmüller type" is a condition on $\mathcal{C}_{|Q|}$ that ensures:

  · each ϕ-orbit in $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ diverges if $\mathrm{ord}(\phi) = \infty$

  · Ψ is injective

$\blacksquare$

<u>Examples</u> of Teichmüller type quivers :

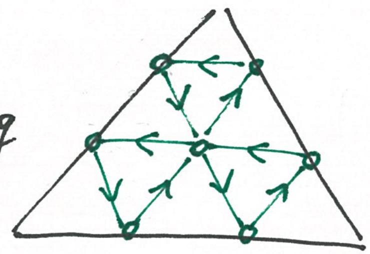
(1) Quivers obtained from (S, Δ).

(2) "Finite type" quivers (= an orientation of a Dynkin diagram)

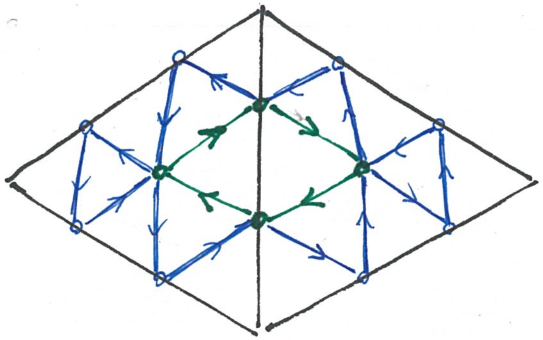
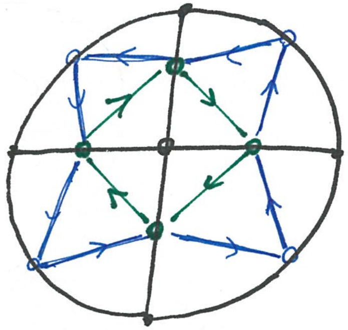
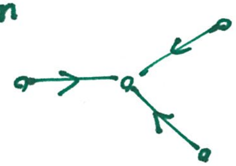
(3) The quivers $I_2(k)$ $(k \geq 2)$.

(4) "SL_3-quivers" on $\underbrace{\overset{A_1}{S_{0,(3)}}, \overset{D_4}{S_{0,(4)}}, \overset{E_7}{S_{0,(5)}}}_{\text{finite type}}$, $S^1_{0,(2)}$

Here the SL_3-quiver Q^3_Δ is obtained by glueing along an ideal triangulation Δ.



<u>Remark</u> $\mathcal{X}_{|Q^3_\Delta|}(\mathbb{R}_{>0}) \cong$ the moduli space of framed SL_3-local systems.

 $=$  $\overset{\text{mutation}}{\simeq}$  D_4

Q^3_Δ on $S_{0,(4)}$ $\qquad$ Q_Δ on $S^1_{0,(4)}$

<u>Theorem</u> (Cerulli-Keller-Labardini-Plamondon '13)

The cluster complex $\mathcal{C}|Q|$ only depends on the mutable part of the initial quiver.