# *Reidemeister torsion for closed hyperbolic three-manifolds*

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# *Outline:*

*Today:* Closed hyperbolic 3-manifolds.

Reidemeister torsion as a topological invariant

- Plan for today:
  1. Reidemeister torsion
  2. Closed hyperbolic three manifolds
  3. Comparison with other invariants
  4. Questions
  5. Lunch

*Tomorrow:* Cusped hyperbolic 3-manifolds of finite volume.

Reidemeister torsion as rational function on the variety of characters

# *Lens spaces (Tietze 1908)*

- $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$

$$(z_1, z_2) \overset{t}{\mapsto} (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i}{p} q} z_2) \qquad p, q \in \mathbb{N} \text{ coprime.}$$

$$L(p, q) = S^3 / \langle t \rangle. \qquad \pi_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$$

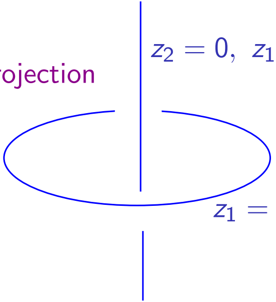
*Question:* For which $q$, $q' \mod p$, $L(p, q) \cong L(p, q')$?

- $q' \equiv -q \mod p$ when $(z_1', z_2') = (z_1, \overline{z}_2)$
- $q' \equiv q^{-1} \mod p$ when $(z_1', z_2') = (z_2, z_1)$

*1935* Franz and Reidemeister: Combinatorial torsion.

Stereographic projection of $S^3$:

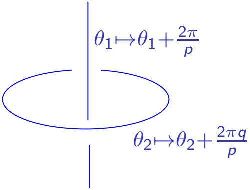
$z_2 = 0,\ z_1 = e^{\theta_1 i},\ \theta_1 \mapsto \theta_1 + \frac{2\pi}{p}$

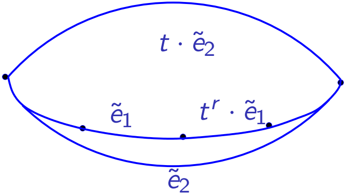


$z_1 = 0,\ z_2 = e^{\theta_2 i},\ \theta_2 \mapsto \theta_2 + \frac{2\pi q}{p}$

# *Lens spaces: Torsion*

$L(p,q) = S^3/\langle t \rangle.$ $\qquad (z_1, z_2) \overset{t}{\mapsto} (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i}{p} q} z_2).$




The lens $\tilde{e}_3 = \{0 \leq \theta_1 \leq \frac{2\pi}{p}\}$ is a fundamental domain for $t$

$$\begin{cases} \partial \tilde{e}_3 = (t-1)\tilde{e}_2 \\ \partial \tilde{e}_2 = (1 + t + \cdots + t^{p-1})\tilde{e}_1 \\ \partial \tilde{e}_1 = (t^r - 1)\tilde{e}_0 \end{cases} \qquad r\,q \equiv 1 \mod p$$

$$t \mapsto \xi \in \mathbb{C}, \quad \begin{matrix} \xi^p = 1 \\ \xi \neq 1 \end{matrix} \quad \begin{cases} \partial \tilde{e}_3 = (\xi - 1)\tilde{e}_2 \\ \partial \tilde{e}_2 = 0 \\ \partial \tilde{e}_1 = (\xi^r - 1)\tilde{e}_0 \end{cases} \qquad H_*(L(p,q), \xi) = 0$$

*Def:* (Reidemeister) $\tau(L(p,q), \xi) := |(\xi - 1)(\xi^r - 1)|$

$$\{\tau(L(p,q),\xi)\}_{\substack{\xi^p = 1 \\ \xi \neq 1}} = \{\tau(L(p,q'),\xi)\}_{\substack{\xi^p = 1 \\ \xi \neq 1}} \Leftrightarrow q' = \pm q^{\pm 1} \mod p$$

## *Combinatorial torsion of a CW-complex*

- $K$ compact CW-complex, $\rho\colon \pi_1 K \to \mathrm{SL_n}(\mathbb{F})$, $\mathbb{F}$=field

*Def:* $C_*(K,\rho) := \mathbb{F}^n_\rho \otimes_{\pi_1 K} C^{CW}_*(\tilde{K},\mathbb{Z})$

$$\left.\begin{array}{l}\{e^i_j\}_j \text{ } i\text{-cells of } K \\ \{v_k\}_k \text{ basis for } \mathbb{F}^n\end{array}\right\} \Rightarrow c_i = \{v_k \otimes \tilde{e}^i_j\}_{j,k} \text{ } \mathbb{F}\text{-basis for } C_i(K,\rho)$$

- $B_i := Im(\partial : C_{i+1}(K,\rho) \to C_i(K,\rho))$
  If $H_*(K,\rho) = 0$ ($\rho$ is acyclic), then

$$0 \to B_i \to C_i(K,\rho) \stackrel{\partial}{\to} B_{i-1} \to 0 \text{ is exact.}$$

$b_i$ $\mathbb{F}$-basis for $B_i$, $\Longrightarrow b_i \sqcup \tilde{b}_{i-1}$ $\mathbb{F}$-basis for $C_i(K,\rho)$.

*Def:* $\tau(K,\rho) := \prod_{i=0}^{n} \det(b_i \sqcup \tilde{b}_{i-1} \mid c_i)^{(-1)^i} \in \mathbb{F}^*/\{\pm 1\}$

- $\tau(K,\rho)$ is a combinatorial invariant (by cellular homeos and subdivision) and invariant of the conjugacy class of $\rho$
- If $H_*(K,\rho) \neq 0$, $\tau(K,\rho,h_*)$ for $h_*$= $\mathbb{F}$-basis of $H_*(K,\rho)$

# *Analytic torsion*

- $M$ smooth closed manifold, $\rho\colon \pi_1(M) \to \mathrm{SL}_n(\mathbb{R})$
  (or $\rho\colon \pi_1(M) \to \mathrm{SL}_n(\mathbb{C}) \subset \mathrm{SL}_{2n}(\mathbb{R})$)
- $E_\rho = \mathbb{R}^n \times_\rho \tilde{M} \to M$ flat vector bundle
  $\Omega^p(M,\rho) = \Gamma(\bigwedge^p T^*M \otimes E_\rho)$ $E_\rho$-valued $p$-forms
  $\Delta^p\colon \Omega^p(M,\rho) \to \Omega^p(M,\rho)$ Laplacian on $E_\rho$-valued $p$-forms

  $\mathrm{Spec}(\Delta^p) = \{\lambda \in \mathbb{R} \mid \exists \omega \in \Omega^p(M;\rho), \Delta^p\omega = \lambda\omega\}$ (discrete)

*Assume* $H^*(M,\rho) = 0$. Then $\mathrm{Spec}(\Delta^p) > 0$

$$\zeta_p(s) = \sum_{\lambda \in \mathrm{Spec}(\Delta^p)} \lambda^{-s} \qquad \text{for } s \in \mathbb{C},\ Re(s) \gg 0$$

$\zeta_p(s)$ extends meromorphically to $s = 0$.

$\zeta_p'(s) = \sum -\lambda^{-s}\log\lambda$, $\Rightarrow$ "$\zeta_p'(0) = -\sum \log\lambda = -\log\det\Delta^p$" !!!

*Def:* (Ray-Singer 1971) Analytic torsion:

$$\tau^{anal}(M,\rho) := \exp\left(\frac{1}{2}\sum_p (-1)^p p\, \zeta_p'(0)\right)$$

# *Cheeger Müller theorem*

- $M$ closed & smooth, $\rho : \pi_1 M \to \mathrm{SL}_n(\mathbb{R})$, $H^*(M, \rho) = 0$
  $\Delta^p : \Omega^p(M; \rho) \to \Omega^p(M; \rho)$ Laplacian on $E_\rho$-valued $p$-forms
  $\tau^{anal}(M, \rho) := \exp\left(\frac{1}{2} \sum_p (-1)^p p\, \zeta_p'(0)\right)$

*Thm:* (Cheeger-Müller) Analytic torsion = Combinatorial torsion

$$\tau^{anal}(M, \rho) = |\tau^{comb}(M, \rho)|$$

- Proved by Jeff Cheeger & Werner Müller for
  $\rho : \pi_1 M \to \mathrm{SO}(n)$ (1978)
- Proved by W. Müller for $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ (1993).

# *Hyperbolic three manifolds*

- $M^3$ closed orientable hyperbolic, $M^3 = \mathbb{H}^3/\Gamma$, $\Gamma < \mathsf{Isom}^+(\mathbb{H}^3)$

$$\mathsf{hol}\colon \pi_1 M^3 \to \Gamma < \mathsf{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$$
$$\mathsf{Ad}_X(y) = X\, y\, X^{-1}, \text{ for } X \in \mathrm{PSL}_2(\mathbb{C}),\ y \in \mathfrak{sl}_2(\mathbb{C})$$

$$\mathsf{Ad}\circ\mathsf{hol}\colon \pi_1 M^3 \to \mathrm{SO}(3,\mathbb{C}) < \mathrm{SL}_3(\mathbb{C})$$

*Thm:* Weil's infinitesimal rigidity (1960's): $H^1(M^3, \mathsf{Ad}\circ\mathsf{hol}) = 0$

*Cor:* $H^*(M^3, \mathsf{Ad}\circ\mathsf{hol}) = H_*(M^3, \mathsf{Ad}\circ\mathsf{hol}) = 0$

*Proof:* $H^0(M^3, \mathsf{Ad}\circ\mathsf{hol}) = \mathfrak{sl}_2(\mathbb{C})^{\pi_1 M^3} = 0$ and duality

*Def:* The torsion of $M^3$ is:

$$\tau(M^3) := \tau(M^3, \mathsf{Ad}\circ\mathsf{hol}) \in \mathbb{C}^*/\{\pm 1\}$$

- By Mostow global rigidity, it is a topological invariant of $M^3$

*Question:* Compare it with other invariants, eg volume.

## *Sequences of thick manifolds*

- Injectivity radius:
  $\text{inj}(M^3) = \frac{1}{2}\inf\{l(\gamma) \mid \gamma \text{ closed geodesic loop in } M^3\}$

*Thm:* (Bergeron-Venkatesh 2010)
Let $M_n^3$ be closed, orientable hyperbolic 3 manifolds.
If $\text{inj}(M_n^3) \to +\infty$ then $\lim_{n\to\infty} \frac{\log|\tau(M_n^3)|}{\text{vol}(M_n^3)} = -\frac{13}{6\pi}$

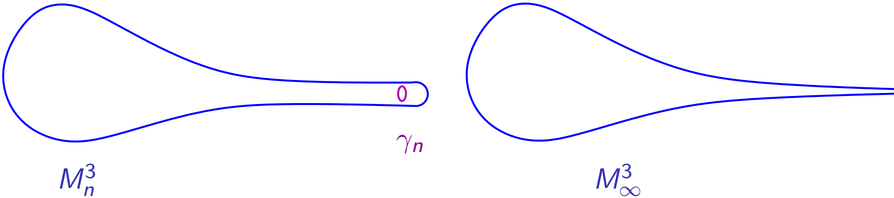
*Cor:* $\tau(M_n^3) \to 0$ as $n \to \infty$.

- Bergeron-Venkatesh Thm uses analytic torsion
- $-\frac{13}{6\pi}$ is an $L^2$ invariant of $\mathbb{H}^3$ and $E_{\text{Ad}}$
- ABBGNRS: generalize to Benjamini-Schramm convergence.

# *Sequences with bounded volume*

*Thm:* (P) $M_n^3$ pairwise different, closed, or., hyperbolic 3-manifolds.
If $\mathrm{vol}(M_n^3) < C$ then $\tau(M_n^3) \to 0$ as $n \to \infty$.

*Proof:* Jørgensen-Thurston: $M_n^3 \to M_\infty^3$ of finite volume with cusps.




- Dehn filling: $M_n^3 \cong \overline{M_\infty^3} \cup N(\gamma_n)$, $\gamma_n$ short geodesic loop, $N(\gamma_n) \cong D^2 \times S^1$ and $\overline{M_\infty^3}$ compact core of $M_\infty^3$.
$\partial D^2 \times \{*\} \sim (p_n, q_n) \in H_1(\partial \overline{M_\infty^3}) \cong \mathbb{Z}^2$, with $p_n^2 + q_n^2 \to \infty$.
- Gluing formula for combinatorial torsion

$$\tau(M_n^3) \sim \tau(\overline{M_\infty^3}, \rho_n, h_*) \frac{1}{p_n + q_n \mathbf{cs}} \tau(N(\gamma_n))$$

where $\mathbf{cs} \in \mathbb{C} - \mathbb{R}$ is the cusp shape, $h_*$ basis for $H_*(\overline{M_\infty^3}, \mathrm{Ad}\,\rho_n)$

# *Sequences with bounded volume*

*Thm:* (P) $M_n^3$ pairwise different, closed, or., hyperbolic 3-manifolds. If $\mathrm{vol}(M_n^3) < C$ then $\tau(M_n^3) \to 0$.

*Proof:* Jørgensen-Thurston: $M_n^3 \to M_\infty^3$ of finite volume with cusps.

- Dehn filling: $M_n^3 \cong \overline{M_\infty^3} \cup N(\gamma_n)$, $\gamma_n$ short geodesic loop, $N(\gamma_n) \cong D^2 \times S^1$ and $\overline{M_\infty^3}$ compact core of $M_\infty^3$. $\partial D^2 \sim (p_n, q_n) \in H_1(\partial \overline{M_\infty^3}) \cong \mathbb{Z}^2$, with $p_n^2 + q_n^2 \to \infty$.
- Gluing formula for combinatorial torsion

$$\tau(M_n^3) \sim \tau(\overline{M_\infty^3}, \rho_n, h_*) \frac{1}{p_n + q_n \mathbf{cs}} \tau(N(\gamma_n))$$

where $\mathbf{cs} \in \mathbb{C} - \mathbb{R}$ is the cusp shape, $h_*$ basis for $H^*(\overline{M_\infty^3}, \rho_n)$

- $\tau(\overline{M_\infty^3}, \rho_n, h_*) \to \tau(\overline{M_\infty^3}, \rho_\infty, h_*)$ bounded,
- $\tau(N(\gamma_n)) = 2 - \cosh(\lambda(\gamma_n))$, $\lambda =$ complex length. $\mathrm{Re}(\lambda(\gamma_n)) = \mathrm{length}(\gamma_n) \to 0$

# *Further sequences of manifolds*

- $M_n^3$ closed or. hyperbolic 3-manifolds, pairwise different
  If $\mathrm{inj}(M_n^3) \to \infty$ or $\mathrm{vol}(M_n^3) < C \;\Rightarrow \tau(M_n^3) \to 0$ as $n \to \infty$

*Question:* Does $\tau(M_n^3) \to 0$ as $n \to \infty$ hold true for any sequence?

- Consider sequences with $\mathrm{inj}(M_n^3) < C$ and $\mathrm{vol}(M_n^3) \to \infty$

*Example:* Iterated mapping tori
$\phi : S \to S$ pseudo Anosov diffeo of a surface $S = S_g$, $g \geq 2$,
$M(\phi) = S \times [0,1]/(x,1) \sim (\phi(x), 0)$.
Look at $M_n^3 = M(\phi^n)$.

*Claim:* $\tau(M(\phi^n)) \to 0$ as $n \to \infty$

## *Iterated mapping tori of a surface diffeo*

- $M(\phi) = S \times [0,1]/(x,1) \sim (\phi(x),0)$, where $\phi\colon S \to S$ p.A.

*Claim:* $\tau(M(\phi^n)) \to 0$ as $n \to \infty$

*Proof:* Use that $\tau(M(\phi)) = 1/\det(d\phi^* - \mathrm{Id})$,

where $\phi^*\colon X(S) \to X(S)$,

$X(S) = \hom(\pi_1 S, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$, and

$d\phi^*\colon T_{[\rho]}X(S) \cong H^1(S, \mathrm{Ad}\circ\rho) \to H^1(S, \mathrm{Ad}\circ\rho)$.

$$\frac{\log|\tau(M(\phi^n))|}{n} = \frac{-\log|\det((d\phi^*)^n - \mathrm{Id})|}{n}$$

$$= \sum_{\lambda\in \mathrm{Spec}(d\phi^*)} \frac{-\log|\lambda^n - 1|}{n} \to \sum_{\substack{\lambda\in \mathrm{Spec}(d\phi^*)\\ |\lambda|>1}} -\log|\lambda| < 0$$

Use $\lambda \in \mathrm{Spec}(d\phi^*) \Rightarrow |\lambda| \neq 1$ (Kapovich) and $\det d\phi^* = 1$. $\square$

*Question:* $\tau(M_n) \to ?$ when $\mathrm{inj}(M_n) < C$ and $\mathrm{vol}(M_n) \to \infty$.

## *Final question*

Assuming $\text{inj}(M_n) < C$ and $\text{vol}(M_n) \to \infty$,

does $\tau(M_n) \to 0$?

Thanks for your attention