Trivial Cocycles, Casson invariant and a Conjecture of Perron

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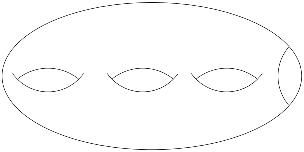
Setting the framework

We start by a standard embedding of a genus $g \geq 3$ surface $\Sigma_{g,1}$ with a disk embedded into the sphere $\mathbb{S}^3$.



This decomposes $\mathbb{S}^3$ into two handlebodies.

Mapping class group

The diagram

$$\begin{array}{ccc} \Sigma_{g,1} & \hookrightarrow & \mathcal{H}_g \\ \big\downarrow & & \big\downarrow \\ -\mathcal{H}_g & \xhookrightarrow{\iota_g} & \mathbb{S}^3 = \mathcal{H}_g \bigcup_{\iota_g} -\mathcal{H}_g \end{array}$$

is a Heegaard splitting of the 3-sphere and gives rise to a diagram of groups:

Mapping class group

$$
\begin{array}{ccc}
\mathcal{M}_{g,1} & \hookleftarrow & \mathcal{A}_{g,1} \\
\uparrow & & \uparrow \\
\mathcal{B}_{g,1} & \hookleftarrow & \mathcal{AB}_{g,1} = \mathcal{A}_{g,1} \cap \mathcal{B}_{g,1}
\end{array}
$$

where

- $\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}(\Sigma_g; \mathrm{rel}. D^2))$ is the "mapping class group".
- $\mathcal{A}_{g,1}$ = subgroup of elements that extend over $\mathcal{H}_g$.
- $\mathcal{B}_{g,1}$ = subgroup of elements that extend over $-\mathcal{H}_g$.
- $\mathcal{AB}_{g,1}$ is the intersection: mapping classes that extend to the whole sphere.

Heegaard splittings of closed manifolds

With this we can parametrize all manifolds.

Definition

Let $\mathcal{V}(3)$ be the set of oriented diffeomorphism classes of closed oriented 3-manifolds.

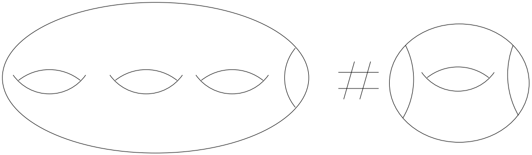
Theorem (Singer, 1953)

The map

$$\begin{array}{rcl} \lim\limits_{g \to \infty} {}_{\mathcal{B}_{g,1}} \backslash \mathcal{M}_{g,1} / {}_{\mathcal{A}_{g,1}} & \longrightarrow & \mathcal{V}(3) \\ \phi & \longmapsto & \mathbb{S}^3_\phi = \mathcal{H}_g \bigcup_{\iota_g \phi} -\mathcal{H}_g \end{array}$$

is a bijection.

The limit is taken along the inclusion maps $\mathcal{M}_{g,1} \hookrightarrow \mathcal{M}_{g+1,1}$ induced by extending a mapping class by the identity:



This is why we need to have this fixed disc.

The Johnson filtration

The mapping class group has a very rich combinatorics:

Theorem (Nielsen)

Let $\pi = \pi_1(\Sigma_{g,1})$. The canonical action of diffeomorphisms on the surface induces an injection:

$$\mathcal{M}_{g,1} \hookrightarrow Aut(\pi).$$

Consider the lower central series of π:

$$\pi \supset [\pi, \pi] \supset [\pi, [\pi, \pi]] \supset \cdots \supset \Gamma_k \supset \ldots$$

$$\Gamma_0 = \pi \quad \Gamma_{k+1} = [\pi, \Gamma_k]$$

The Johnson filtration

The action of $\mathcal{M}_{g,1}$ on π respects this filtration

$$\pi \supset [\pi,\pi] \supset [\pi,[\pi,\pi]] \supset \cdots \supset \Gamma_k \supset \ldots$$

hence induces maps $\forall k \geq 0$

$$\mathcal{M}_{g,1} \xrightarrow{\tau_k} Aut(\pi/\Gamma_{k+1})$$

For instance $\tau_0 = H_1(-)$
Let

$$\mathcal{M}_{g,1}(k+1) = \ker \tau_k$$

The Johnson filtration

This gives a descending and separated filtration of the mapping class group:

$$\mathcal{M}_{g,1} \supset \mathcal{M}_{g,1}(1) \supset \mathcal{M}_{g,1}(2) \ldots \quad \bigcap_{k=1}^{\infty} \mathcal{M}_g(k) = \{Id\}$$

This is the Johnsons filtration, and the quotients $\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k)$ have been the object of much study (S. Morita, R. Hain, many others).

Two questions with partial answers

Recall Singer's and Nielsen's theorem:

Theorem (Singer, 1953)

The map

$$\begin{array}{rcl} \lim\limits_{g\to\infty} {}_{\mathcal{B}_{g,1}}\backslash \mathcal{M}_{g,1} / {}_{\mathcal{A}_{g,1}} & \longrightarrow & \mathcal{V}(3) \\ \phi & \longmapsto & \mathbb{S}^3_\phi = \mathcal{H}_g \bigcup_{\iota_g \phi} -\mathcal{H}_g \end{array}$$

is a bijection.

Theorem (Nielsen)

Let $\pi = \pi_1(\Sigma_{g,1})$. The canonical action of diffeomorphisms on the surface induces an injection:

$$\mathcal{M}_{g,1} \hookrightarrow Aut(\pi).$$

If you know the action of a mapping class on the fundamental group, then you "know" the manifold it builds.

First question

We have an increasingly accurate series of approximations of the action on the fundamental groups

$$\begin{array}{ccc} \mathcal{M}_{g,1} & \longrightarrow & Aut\pi \\ & \searrow^{\tau_k} & \downarrow \\ & & Aut(\pi/\Gamma_{k+1}) \end{array}$$

Question

What can you say about the manifold $\mathbb{S}^3_\phi$ if you know the action of ϕ on π up to $k+1$-commutators? i.e. you only know $\tau_k(\phi)$?

Pointing towards an answer?

Some easy cases:

1. By Mayer-Viettoris, if you know $\tau_0(\phi) = H_1(\phi; \mathbb{Z})$ you know the cohomology of $\mathbb{S}^3_\phi$ as a group.
2. To know the ring structure you only need $\tau_1(\phi)$ i.e the action on $\pi/[\pi, [\pi, \pi]]$ (Stallings).

Cochran,Gerges, Orr (2001)

Definition

Two closed 3-manifolds M_0 and M_1 are k-surgery equivalent if there exists a sequence $M_0 = X_0 \dots X_2 \dots X_m = X_1$ such that

- X_{j+1} is obtained from X_j by $\pm\frac{1}{q_j}$ surgery along a curve $\gamma_j \in \Gamma_k(\pi_1(X_i))$

Theorem (Cochran,Gerges,Orr (2001))

The following are equivalent:

1. *M_0 and M_1 are k-equivalent*
2. *$\exists \phi : \pi_1(M_0)/\Gamma_k(M_0) \xrightarrow{\sim} \pi_1(M_1)/\Gamma_k(M_1)$ such that $\phi([M_0]) = [M_1]$ where $[M_i]$ is the image in $H_3(\pi_1(M_i)/\Gamma_k(M_i);\mathbb{Z})$ of the fundamental class of M along the canonical map $f_i : M_i \to K(\pi_1(M_i)/\Gamma_k(M_i), 1)$.*

1. Question: how is k-equvalence related to equality under τ_k.

$$\mathbb{S}^3_\phi \sim_k \mathbb{S}^3_\psi \overset{?}{\Longleftrightarrow} \tau_k(\phi) = \tau_k(\psi)$$

2. For $k = 2$ this is true (Cochran,Gerges,Orr)

Second Question

Question

Assume know that $\phi \in \mathcal{M}_{g,1}(k)$, i.e. the action on π up to k-commutators is trivial. What is $\mathbb{S}^3_\phi$?

By Mayer-Viettoris, $\mathbb{S}^3_\phi$ is an integral homology sphere.

$$\text{Let } \mathcal{S}(3) = \{M \mid H_*(M;\mathbb{Z}) = H_*(\mathbb{S}^3;\mathbb{Z})\}.$$

$$\text{and } \mathcal{S}(3)_k = \{\mathbb{S}^3_\phi \mid \phi \in \lim_g \mathcal{M}_{g,1}(k)\}$$

Known cases

1. For $k = 1$ $\mathcal{S}(3)_1 = \mathcal{S}(3)$ (exercise in Mayer-Viettoris)
2. For $k = 2$ $\mathcal{S}(3)_2 = \mathcal{S}(3)$ (Prof. Morita)
3. For $k = 3$ $\mathcal{S}(3)_3 = \mathcal{S}(3)$ (W. P. and Massuyeau-Meilhan)
4. For $k \geq 4$, unknown. Maybe yes?

Trivial cocycles and Casson invariant

One of the difficulites in the above question is to understand the restriction of the doble coset relation ${}_{\mathcal{B}_{g,1}}\backslash\mathcal{M}_{g,1}/_{\mathcal{A}_{g,1}}$ to the groups $\mathcal{M}_{g,1}(k)$. Denote by $\approx$ this equivalence relation.

Proposition

$\forall\phi,\psi\in\mathcal{M}_{g,1}(1)$

$$\phi\approx\psi\Leftrightarrow\begin{cases}\exists\mu\in\mathcal{AB}_{g,1} such\ that\\ \phi=\mu\psi\mu^{-1}\in{}_{\mathcal{B}_{g,1}(1)}\backslash\mathcal{M}_{g,1}(1)/_{\mathcal{A}_{g,1}(1)}\end{cases}$$

where $\mathcal{A}_{g,1}(k)=\mathcal{A}_{g,1}\cap\mathcal{M}_{g,1}(k)$ *and similarly for* $\mathcal{B}_{g,1}(k)$.

$\approx$ is double class in $\mathcal{M}_{g,1}(1)+$ coinvariants under the action of $\mathcal{AB}_{g,1}$.

From invariants to trivial cocycles

Let $F : \mathcal{S}(3) \to A$ be an invariant, where A is a group without 2-torsion. This is the same as a familly of functions

$$\begin{array}{ccc} & \lim_{g\to\infty} \mathcal{M}_{g,1}(1)/\approx & \\ \nearrow & & \searrow^{F} \\ \mathcal{M}_{g,1}(1) & \xrightarrow[F_g]{} & A \end{array}$$

$\forall \phi, \psi \in \mathcal{M}_{g,1}(1)$ let $C_g(\phi,\psi) = F_g(\phi\psi) - F_g(\phi) - F_g(\psi)$
This is a trivialized 2-cocycle on $\mathcal{M}_{g,1}$

Properties of C_g

$$C_g(\phi, \psi) = F_g(\phi\psi) - F_g(\phi) - F_g(\psi)$$

Because F_g is constant on the equivalence classes, C_g has nice properties.

1. C_{g+1} restricted to $\mathcal{M}_{g,1}(1)$ is C_g.
2. C_g is invariant under conjugation by $\mathcal{AB}_{g,1}$
   $C_g(\mu\phi\mu^{-1}, \mu\psi\mu^{-1}) = C_g(\phi, \psi)$
3. $C_g = 0$ on $\mathcal{M}_{g,1}(1) \times \mathcal{A}_{g,1}(1) \cup \mathcal{B}_{g,1}(1) \times \mathcal{M}_{g,1}(1)$
4. $C_g \neq 0$, unless $F = 0$, equivalently C_g is associated to a unique F.

Observe that C_g measures the defect to be a homomorphism. It can alos be seen as a kind of "surgery instruction".

From cocycles to invariants

Theorem (W.P.)

Let A be an abelian group wihtout 2-torsion. Let $(C_g)_{g\geq 3}$ be a familiy of 2-cocycles on $\mathcal{M}_{g,1}(1)$ such that

1. C_{g+1} *restricted to* $\mathcal{M}_{g,1}(1)$ *is* C_g.
2. C_g *is invariant under conjugation by* $\mathcal{AB}_{g,1}$
   $C_g(\mu\phi\mu^{-1}, \mu\psi\mu^{-1}) = C_g(\phi, \psi)$.
3. $C_g = 0$ *on* $\mathcal{M}_{g,1}(1) \times \mathcal{A}_{g,1}(1) \cup \mathcal{B}_{g,1}(1) \times \mathcal{M}_{g,1}(1)$.
4. $[C_g] = 0$ *in* $H^2(\mathcal{M}_{g,1}(1); A)$.
5. *The associated torsor* $\rho_{C_g} \in H^1(\mathcal{AB}_{g,1}; Hom(\mathcal{M}_{g,1}(1), A))$ *is* 0.

Then C_g is the defect of a unique invariant F with values in A, where F_g is the unique $\mathcal{AB}_{g,1}$-invariant trivialization of C_g.

Algebraic construction of the cvAsson invariant

- The Casson invariant $\lambda : \mathcal{S}(3) \to \mathbb{Z}$ of $M \in \mathcal{S}(3)$ essentially counts the number of representations of $\pi_1(M)$ in $SU(2)$.
- The Casson invariant is determined by surgery properties.

Let $H = H_1(\Sigma_g; \mathbb{Z})$ and $\omega : H \times H \to \mathbb{Z}$ the (symplectic) intersection form.

- The embedding $\Sigma_g \hookrightarrow \mathbb{S}^3$ determines two transverse lagrangians $A \oplus B = H$,
- The abelianization of $\mathcal{M}_{g,1}(1) \simeq \Lambda^3 H \oplus$ 2-torsion. Let $\tau_1 : \mathcal{M}_{g,1}(1) \to \Lambda^3 H$.
- View the intersection form as a map $\omega : A \times B \to \mathbb{Z}$. It induces $\Lambda^3\omega : \Lambda^3 A \times \Lambda^3 B \to \mathbb{Z}$.
- On $\Lambda^3 H = \Lambda^3 A \oplus W_{AB} \oplus \Lambda^3 B$ consider the bilinear form (i.e. 2-cocycle!)

$$2J_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Lambda^3\omega & 0 & 0 \end{pmatrix}$$

Then one can apply the previous theorem to the pull back $\tau_1^*(2J_g)$ and from the surgery formula recognize the associated invariant as being the Casson invariant.
Otherwise said the relation:

$$\forall \phi, \psi \in \mathcal{M}_{g,1}(1) \quad \lambda(\mathbb{S}^3_{\phi\psi}) - \lambda(\mathbb{S}^3_{\phi}) - \lambda(\mathbb{S}^3_{\psi}) = 2J_g(\tau_1(\phi), \tau_1(\psi))$$

defines the Casson invariant.

Question

Find the cocycles associated to other invariants, for instance those defined through Ohtsuki's theory of finite type invariants.

Perron's conjecture

joint with R. Riba

Let $p \neq 2$ be a prime number. Let

$$\mathcal{M}_{g,1}[p] = \ker\left(\mathcal{M}_{g,1} \xrightarrow{H_1} Sp(2g,\mathbb{Z}) \twoheadrightarrow Sp(2g,\mathbb{Z}/p\mathbb{Z})\right)$$

If $\phi \in \mathcal{M}_{g,1}[p]$ then the associated $\mathbb{S}^3_\phi$ is a mod-p homology sphere.

$$\mathcal{S}(3,p) = \{M \in \mathcal{V}(3) \mid H_*(M;\mathbb{Z}/p\mathbb{Z}) = H_*(\mathbb{S}^3;\mathbb{Z}/p\mathbb{Z})\}$$

Proposition

There is a bijection

$$\lim_{g\to\infty} \mathcal{M}_{g,1}[p]/\approx \longrightarrow \mathcal{V}(3)$$

where $\approx$ is as before double coset + conjugation by $\mathcal{AB}_{g,1}$

Perro's results and statement

- Every element $\phi \in \mathcal{M}_{g,1}[p]$ can be written as a product

$$\phi = f_\phi T_{\gamma_1}^{\pm p} T_{\gamma_2}^{\pm p} \cdots T_{\gamma_n}^{\pm p}$$

where the γ_i are simple closed curves on Σ_g and $f_\phi \in \mathcal{M}_{g,1}[p]$.

Conjecture

Conjecture: If λ denotes the Casson invariant, then $\lambda(\mathbb{S}^3_{f_\phi})$ mod p is an invariant of $\mathcal{S}^3_\phi$.

The whole setting for understanding invariants of integral homology spheres works for mod-p homology spheres.

Theorem (W.P.)

Let $(C_g)_{g\geq 3}$ be a familiy of 2-cocycles with values in $\mathbb{Z}/p\mathbb{Z}$ on $\mathcal{M}_{g,1}[p]$ such that

1. C_{g+1} *restricted to* $\mathcal{M}_{g,1}[p]$ *is* C_g.
2. C_g *is invariant under conjugation by* $\mathcal{AB}_{g,1}$
   $C_g(\mu\phi\mu^{-1}, \mu\psi\mu^{-1}) = C_g(\phi, \psi)$.
3. $C_g = 0$ *on* $\mathcal{M}_{g,1}[p] \times \mathcal{A}_{g,1}[p] \cup \mathcal{B}_{g,1}[p] \times \mathcal{M}_{g,1}[p]$.
4. $[C_g] = 0$ *in* $H^2(\mathcal{M}_{g,1}[p]; \mathbb{Z}/p\mathbb{Z})$.
5. *The associated torsor* $\rho_{C_g} \in H^1(\mathcal{AB}_{g,1}; Hom(\mathcal{M}_{g,1}[p], \mathbb{Z}/p\mathbb{Z}))$ *is* 0.

Then C_g is the defect of p different invariants F with values in $\mathbb{Z}/p\mathbb{Z}$, the associated p functions F_g are the unique $\mathcal{AB}_{g,1}$-invariant trivializations of C_g.

Consider the cocycle on $\Lambda^3 H$ defining the Casson invariant and reduce it mod p.

$$2J_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Lambda^3\omega & 0 & 0 \end{pmatrix}$$

Then there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_{g,1}(1) & \hookrightarrow & \mathcal{M}_{g,1}[p] \\ \downarrow \tau_1 & \swarrow \exists! \widetilde{\tau}_1 & \\ \Lambda^3 H \ \mathrm{mod} - p & & \end{array}$$

Then apply the theorem to $\widetilde{\tau}_1^*(2J_g)$. Under scrutiny: triviality of the cocycle.

Thank you for your attention.