Non-Kähler complex structures on $\mathbb{R}^4$

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- Problem and Motivation
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Kähler and non-Kähler

Definition

A complex mfd (M, J) is said to be Kähler if there exists a symplectic form ω compatible with J, i.e.,

1. $\omega(u, Ju) > 0$ for any $u \neq 0 \in TM$,
2. $\omega(u, v) = \omega(Ju, Jv)$ for any $u, v \in TM$.

Projective varieties, Calabi-Yau manifolds, and Stein manifolds are all Kähler.
Hopf manifolds, Calabi-Eckmann manifolds, and Kodaira-Thurston manifolds are non-Kähler.

Compact complex surfaces

Compact complex surfaces are classified into seven classes:
(I) $\mathbb{C}P^2$ or ruled surfaces, (II) K3 surfaces, (III) complex tori,
(IV) Kähler elliptic surfaces, (V) alg surfaces of general type,
(VI) non-Kähler elliptic surfaces, (VII) surfaces with $b_1 = 1$.

Theorem (Miyaoka, Siu)

A compact complex surface is Kähler iff its first Betti number b_1 is even.

(I) – (V) are Kähler and (VI), (VII) are non-Kähler.

Our problem

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- If $n \geq 3$, "Yes" (Calabi-Eckmann 1953).

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Is there any non-Kähler complex structure on $\mathbb{R}^{2n}$?

- If $n = 1$, the answer is clearly "No".
- If $n \geq 3$, "Yes" (Calabi-Eckmann 1953).
- If $n = 2$, "Yes" (Di Scala-K-Zuddas 2015).

Calabi-Eckmann's construction

$H_1 : S^{2p+1} \to \mathbb{C}P^p$, $H_2 : S^{2q+1} \to \mathbb{C}P^q$: the Hopf fibrations.
$H_1 \times H_2 : S^{2p+1} \times S^{2q+1} \to \mathbb{C}P^p \times \mathbb{C}P^q$ is a T^2-bundle.
The Calabi-Eckmann manifold $M_{p,q}(\tau)$ is a complex mfd diffeo to $S^{2p+1} \times S^{2q+1}$ s.t. $H_1 \times H_2$ is a holomorphic torus bundle (τ is the modulus of a fiber torus).
$E_{p,q}(\tau) := (S^{2p+1} \setminus \{p_0\}) \times (S^{2q+1} \setminus \{q_0\}) \subset M_{p,q}(\tau)$.
If $p > 0$ and $q > 0$, then it contains holomorphic tori.

So, it is diffeo to $\mathbb{R}^{2p+2q+2}$ and non-Kähler.

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- This doesn't work if $p = 0$ or $q = 0$.

Non-Kählerness and holomorphic curves

Lemma (1)

If a complex manifold $(\mathbb{R}^{2n}, J)$ *contains a compact holomorphic curve* C*, then it is non-Kähler.*

Proof.

Suppose it is Kähler. Then, there is a symp form ω compatible with J. Then, $\int_C \omega > 0$. Hence, C represents a nontrivial 2nd homology. This is a contradiction. □

Main Theorem

Let $P = \left\{0 < \rho_1 < 1, 1 < \rho_2 < \rho_1^{-1}\right\} \subset \mathbb{R}^2$.

Theorem (D-K-Z, to appear in Geom.Topol.)

For any $(\rho_1, \rho_2) \in P$, there are a complex manifold $E(\rho_1, \rho_2)$ diffeomorphic to $\mathbb{R}^4$ and a surjective holomorphic map $f : E(\rho_1, \rho_2) \to \mathbb{C}P^1$ such that the only singular fiber $f^{-1}(0)$ is an immersed holomorphic sphere with one node, and the other fibers are either holomorphic tori or annuli. Moreover, $E(\rho_1, \rho_2)$ and $E(\rho_1', \rho_2')$ are distinct if $(\rho_1, \rho_2) \neq (\rho_1', \rho_2')$.

The Matsumoto-Fukaya fibration

$f_{MF} : S^4 \to \mathbb{C}P^1$ is a genus-1 achiral Lefschetz fibration with only two singularities of opposite signs.

F_1: the fiber with the positive singularity $\big((z_1, z_2) \mapsto z_1 z_2\big)$

F_2: the fiber with the negative singularity $\big((z_1, z_2) \mapsto z_1 \bar{z}_2\big)$

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- $N_1 \cup (N_2 \backslash X) \cong \mathbb{R}^4$ (X is a nbd of $-$ sing),
- Topologically, $E(\rho_1, \rho_2)$ is $N_1 \cup (N_2 \backslash X)$ and f is the restriction of f_{MF}.

The Matsumoto-Fukaya fibration 2

Originally, it is constructed by taking the composition of the Hopf fibration $H : S^3 \to \mathbb{C}P^1$ and its suspension $\Sigma H : S^4 \to S^3$. $f_{MF} = H \circ \Sigma H$.

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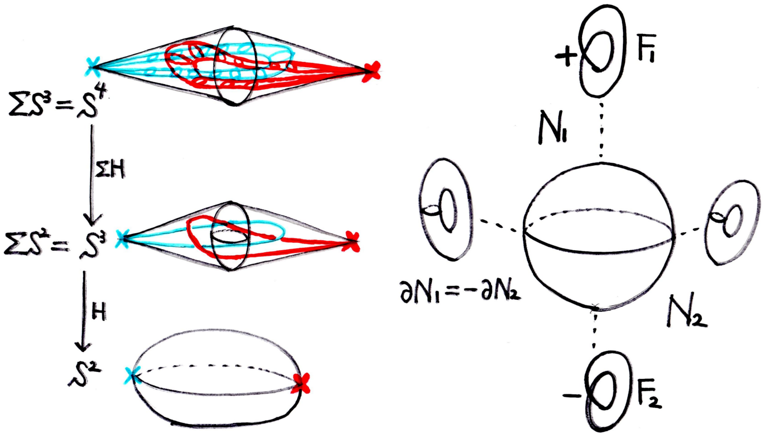
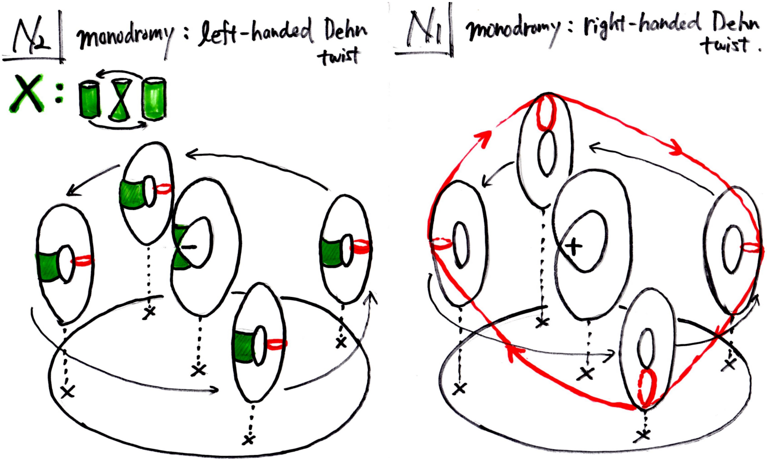
- The two pinched points correspond to the two singularities (in the next page).
- How to glue ∂N_2 to ∂N_1 is as the pictures in the page after next.

The Matsumoto-Fukaya fibration 3



Gluing N_1 and N_2



Kirby diagrams

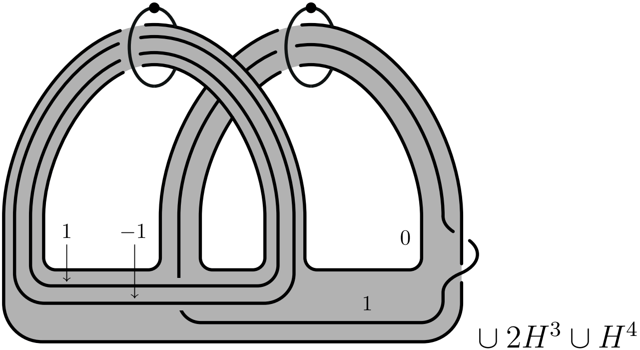


Figure: The Matsumoto-Fukaya fibration on S^4.

Kirby diagrams 2

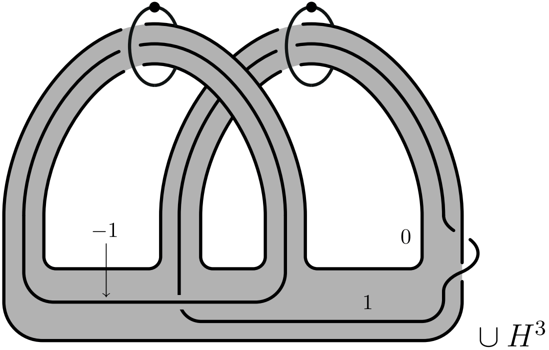


Figure: The map f on $S^4 \setminus X \cong \mathbb{R}^4$.

Key Lemma

Let A denote an annulus.

Lemma (2)

Let us glue $A \times D^2$ *to* N_1 *so that for each* $t \in \partial D^2 = S^1$*, the annulus* $A \times \{t\}$ *embeds in the fiber torus* $f^{-1}(t)$ *as a thickened meridian, and that it rotates in the longitude direction once as* $t \in S^1$ *rotates once. Then, the resultant manifold is diffeomorphic to* $\mathbb{R}^4$*.*

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- **This topological lemma gives us the "blueprint".**
- **We will realize this gluing by complex manifolds!**

Holomorphic models

$\Delta(r) := \{|z| < r\} \subset \mathbb{C}$,
$\Delta(r_1, r_2) := \{r_1 < |z| < r_2\} \subset \mathbb{C}$.
$N_1 \rightsquigarrow W$: Kodaira's holomorphic model,
$N_2 \backslash X \rightsquigarrow \Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.
The elliptic fibration

$$\pi : \mathbb{C}^* \times \Delta(0, \rho_1)/\mathbb{Z} \to \Delta(0, \rho_1),$$

where $n \cdot (z, w) = (zw^n, w)$, extends to a singular elliptic fibration $f_1 : W \to \Delta(\rho_1)$, whose singular fiber is type I_1.

Gluing domains in the two pieces

The gluing domain in the product part is
$V_2 := \Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \subset \Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.

The gluing domain $V_1 \subset W$ is defined as follows.
Put $\varphi(w) = \exp\left(\frac{1}{4\pi i}(\log w)^2 - \frac{1}{2}\log w\right)$.

$$V_1 := \{[z\varphi(w), w] \mid z \in \Delta(1, \rho_2), w \in \Delta(\rho_0, \rho_1)\}.$$

Then, $V_1 \cong \Delta(1, \rho_2) \times \Delta(\rho_0, \rho_1)$.

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- $\varphi(re^{i(\theta+2\pi)}) = re^{i\theta}\varphi(re^{i\theta}) = w\varphi(w)$.

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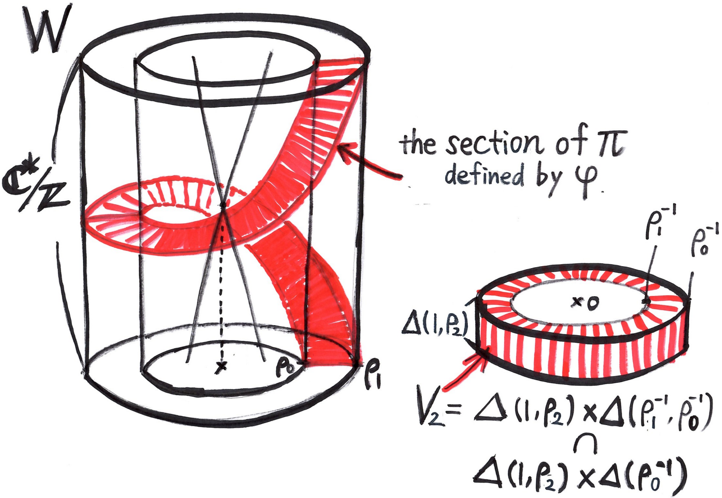

Gluing the two pieces

By the biholomorphism between the gluing domains
$\Phi : V_2 \to V_1;\ (z, w^{-1}) \mapsto [(z\varphi(w), w)]$,
we obtain a complex manifold

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- $\Delta(\rho_1)$ and $\Delta(\rho_0^{-1})$ are glued to become $\mathbb{C}P^1$.
- f is defined to be $f_1 : W \to \Delta(\rho_1)$ on W, and the 2nd projection on $\Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$.

Classification of holomorphic curves

Lemma (3)

Any compact holomorphic curve in $E(\rho_1, \rho_2)$ is a compact fiber of the map $f : E(\rho_1, \rho_2) \to \mathbb{C}P^1$.

Proof.

Let $i : C \to E(\rho_1, \rho_2)$ be a compact holomorphic curve. The composition $f \circ i : C \to \mathbb{C}P^1$ is a holomorphic map between compact Riemann surfaces. It is either a brached covering or a constant map. Since $E(\rho_1, \rho_2)$ is contractible, $f \circ i$ is homotopic to a constant map. So, it is a constant map. □

Properties of $E(\rho_1, \rho_2)$

If $E(\rho_1, \rho_2) \cong E(\rho_1', \rho_2')$, then $(\rho_1, \rho_2) = (\rho_1', \rho_2')$.

Proof.

Let $\Psi \colon E(\rho_1, \rho_2) \to E(\rho_1', \rho_2')$ be a biholomorphism.
Since Ψ sends a compact curve to a compact curve, it is a fiberwise biholomorphism on W. Looking at the moduli of elliptic fibers, the base map $\Delta(\rho_1) \to \Delta(\rho_1')$ must be an identity. We obtain $\rho_1 = \rho_1'$.

By analyticity, it is fiberwise also on the whole $E(\rho_1, \rho_2)$.

Since $\Delta(1, \rho_2) \cong \Delta(1, \rho_2')$, we have $\rho_2 = \rho_2'$. □

In particular, there are uncountable non-Kähler complex structures on $\mathbb{R}^4$.

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- $f^* : \mathrm{Pic}(\mathbb{C}P^1) \to \mathrm{Pic}(E(\rho_1, \rho_2))$ is injective.

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- $E(\rho_1, \rho_2) \times \mathbb{C}^{n-2}$ give uncountably many non-Kähler complex structures on $\mathbb{R}^{2n}$ $(n \geq 3)$.

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- It cannot be holomorphically embedded in any compact complex surface.

Noncompact non-Kähler complex surfaces

Theorem

Any connected open oriented 4-manifold admits uncountable non-Kähler complex structures.

It is the consequence of a simple application of our complex $\mathbb{R}^4$ and Phillips' theorem.

Theorem (Phillips)

Let M be an open manifold. Then, the map

$d\colon \mathrm{Sub}(M, V) \to \mathrm{Epi}(TM, TV);\ f \mapsto df$

is a weak homotopy equivalence.

Thank you for your attention!