Skein algebras and mapping class groups on oriented surfaces

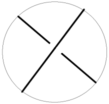
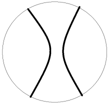
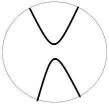
Shunsuke TSUJI

The University of Tokyo

Kauffman bracket

D: link diagram in $\mathbb{R}^2 \subset \mathbb{R}^3$

$\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ is defined by

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$\cdots$ skein relation

$\langle D' \sqcup \mathcal{O} \rangle = (-A^2 - A^{-2}) \langle D' \rangle$ ··trivial knot relation

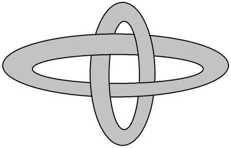
$\langle \emptyset \rangle = 1$

Here $\mathcal{O}$: trivial knot, $\emptyset$: empty link.

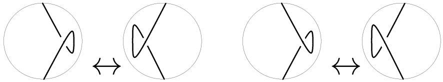
$\rightsquigarrow \langle D \rangle$: invariant of framed links

Framed links

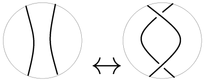
A framed link is an isotopy type of an image of an embedding of a disjoint union of annuli into $\mathbb{R}^3$.



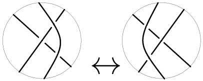
RI



RII



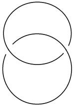
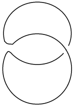
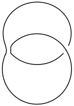
RIII

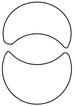
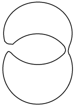


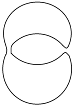
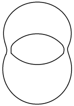
In other words,

$$(\text{the set of framed links}) = \frac{(\text{the set of link diagrams})}{\text{RI,RII,RIII}}.$$

An example of Kauffman bracket

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= A^2\langle$  $\rangle + \langle$  $\rangle$

$+\langle$  $\rangle + A^{-2}\langle$  $\rangle$

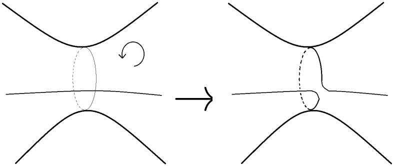
$$= A^2(-A^2 - A^{-2})^2 + (-A^2 - A^{-2})$$
$$+(-A^2 - A^{-2}) + A^{-2}(-A^2 - A^{-2})^2$$
$$= (A^4 + A^{-4})(A^2 + A^{-2})$$

Mpping class groups

Σ: compact connected surface
$\mathrm{Diff}^+(\Sigma, \partial\Sigma)$: the set of orientation preserving diffeomorphisms which fixes $\partial\Sigma$ point wise
$\mathcal{M}(\Sigma) \stackrel{\mathrm{def.}}{=} \mathrm{Diff}^+(\Sigma, \partial\Sigma)/$ (isotopy rel. $\partial\Sigma$)
$\mathcal{M}(\Sigma)$: mapping class group of Σ

Dehn twists

c: s.c.c.(simple closed curve) in Σ
$t_c \in \mathcal{M}(\Sigma)$: (right hand) Dehn twist along c



Fact (Dehn-Lickorish)

$\mathcal{M}(\Sigma)$ is generated by Dehn twists.

abstract

- We will establish some explicit relationship between the study of Kauffman bracket and the study of mapping class groups.
- We study the action of mapping class groups on Kauffman bracket skein algebras and and Kauffman bracket skein modules.
- We need ‘the logarithms‘ of Dehn twists.
- We need ‘completions‘ of skein algebras and skein modules.

Tangles in $\Sigma \times I$

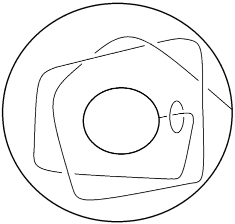
Σ: oriented compact connected surface

$J = \{P_1, P_2, \cdots, P_{2k}\} \subset \partial\Sigma$, $I = [0, 1]$

$\mathcal{T}(\Sigma, J)$: the set of unoriented framed tangle in $\Sigma \times I$ with the set of base points $J \times \{\frac{1}{2}\}$

$\mathcal{T}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{T}(\Sigma, \emptyset)$

(ex) $S^1 = \mathbb{R}/\mathbb{Z}$, $\mathcal{T}(S^1 \times I, \{(0, 0), (0, 1)\})$



Skein algebras and Skein modules

$$\mathcal{S}(\Sigma, J) \stackrel{\text{def.}}{=} \frac{\mathbb{Q}[A, A^{-1}]\mathcal{T}(\Sigma, J)}{\text{(skein relation, trivial knot relation)}}$$

$$\mathcal{S}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{S}(\Sigma, \emptyset)$$

$T \in \mathcal{T}(\Sigma, J)$

$\langle T \rangle$: the element of $\mathcal{S}(\Sigma, J)$ represented by T

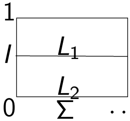
$\xi \in \mathcal{M}(\Sigma)$

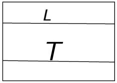
$$\xi\langle T \rangle \stackrel{\text{def.}}{=} \langle \xi \times \mathrm{id}_I(T) \rangle$$

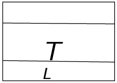
$\rightsquigarrow \mathcal{M}(\Sigma) \curvearrowright \mathcal{S}(\Sigma, J)$

The product and the action

$L, L_1, L_2 \in \mathcal{T}(\Sigma),\ T \in \mathcal{T}(\Sigma, J)$

$L_1L_2 \overset{\text{def.}}{=}$ 

$\cdots(1)$ $LT \overset{\text{def.}}{=}$ 

$\cdots(2)$ $TL \overset{\text{def.}}{=}$ 

$\cdots(3)$

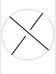
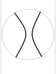
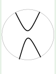
$\mathcal{S}(\Sigma) \times \mathcal{S}(\Sigma) \to \mathcal{S}(\Sigma)$ product by (1)
$\mathcal{S}(\Sigma) \curvearrowright \mathcal{S}(\Sigma, J)$ left action by (2)
$\mathcal{S}(\Sigma, J) \curvearrowleft \mathcal{S}(\Sigma)$ right action by (3)

Lie algebra and Lie action

Remark

$\mathcal{S}(\Sigma, J)$ is a free $\mathbb{Q}[A, A^{-1}]$-module.

Remark

$\langle$  $\rangle - \langle$  $\rangle = (A - A^{-1})(\langle$  $\rangle - \langle$  $\rangle$

Definition

For $x, y \in \mathcal{S}(\Sigma)$, $[x, y] \stackrel{\text{def.}}{=} \frac{1}{-A+A^{-1}}(xy - yx)$.

$\rightsquigarrow (\mathcal{S}(\Sigma), [\,,\,])$: Lie algebra

Definition

For $x \in \mathcal{S}(\Sigma)$ *and* $z \in \mathcal{S}(\Sigma)$, $\sigma(x)(z) \stackrel{\text{def.}}{=} \frac{1}{-A+A^{-1}}(xz - zx)$.

$\rightsquigarrow \mathcal{S}(\Sigma, J)$: $\mathcal{S}(\Sigma)$-module with σ

The group ring of the 1st homology

$H \stackrel{\text{def.}}{=} H_1(\Sigma)$

$\mathbb{Q}H$: the group ring of H over $\mathbb{Q}$

$\mu : H \times H \to \mathbb{Z}$: the intersection form of H

$$
\begin{aligned}
[\ ,\ ] : \mathbb{Q}H \times \mathbb{Q}H &\to \mathbb{Q}H \\
(a, b) &\mapsto \mu(a, b)ab
\end{aligned}
$$

Here $a, b \in H$.

$\rightsquigarrow (\mathbb{Q}H, [,])$: Lie algebra

(furthermore Poisson algebra)

A homomorphism $\epsilon' : \mathcal{S}(\Sigma) \to \mathbb{Q}H$

$\mathbb{Q}$-algebra homomorphism $\epsilon' : \mathcal{S}(\Sigma) \to \mathbb{Q}H$ is defined by

- $\epsilon'(A) = -1$,
- $\epsilon'(\langle L \rangle) = \prod_{i=1}^{m}(-[l_i] - [l_i]^{-1})$.

Here $L = l_1 \cup l_2 \cup \cdots \cup l_m$ (l_i: component of L).

Proposition

- *ϵ' is well-defined.*
- *ϵ' is a Lie algebra homomorphism. In other words, for $x, y \in \mathcal{S}(\Sigma)$, $[\epsilon'(x), \epsilon'(y)] = \epsilon'([x, y])$.*

An augmentation map ϵ

Σ: compact connected oriented surface with non-empty boundary

$* \in \partial\Sigma$, $p_1 : \Sigma \times I \to \Sigma, (x, t) \mapsto x$

$\epsilon : \mathcal{S}(\Sigma) \to \mathbb{Q}$ is defined by

- $\epsilon(A) = -1$,
- $\epsilon(\langle L \rangle) = (-2)^{|L|}$.

Here $|L|$ is the number of components of L.

$\rightsquigarrow \epsilon$ is well-defined.

Goldman Lie algebra 1/3

$\pi \overset{\text{def.}}{=} \pi_1(\Sigma, *)$, $\hat{\pi} \overset{\text{def.}}{=} \pi/(\text{conj.})$

$|\cdot| : \mathbb{Q}\pi \to \mathbb{Q}\hat{\pi}$, the quotient map

=the forgetfulmap of the base point

$\pi_{\square} \overset{\text{def.}}{=} \pi/(|x| \sim |x^{-1}|)$

$|\cdot|_{\square} : \mathbb{Q}\pi \to \mathbb{Q}\pi_{\square}$, the quotient map

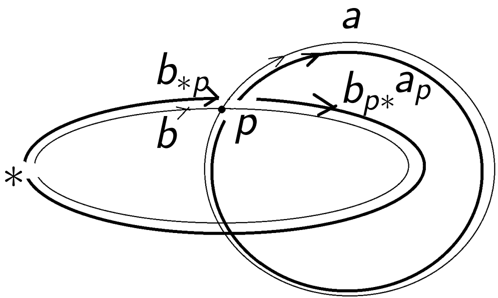
=the forgetfulmap of the base point and orientations

Goldman Lie algebra 2/3

$a \in \hat{\pi}$, $b \in \pi$ in general positon

$\sigma(a)(b) \stackrel{\text{def.}}{=} \sum_{p \in a \cap b} \varepsilon(p, a, b) b_{*p} a_p b_{p*}$

$\varepsilon(p, a, b)$: local intersection number of a and b at p



$[a, |b|] \stackrel{\text{def.}}{=} |\sigma(a)(b)|$

Goldman Lie algebra 3/3

$a \in \pi$, $b \in \pi$
$\sigma(|a|_\square)(b) \stackrel{\text{def.}}{=} \sigma(|a|)(b) + \sigma(|a^{-1}|)(b)$
$[|a|_\square, |b|_\square] \stackrel{\text{def.}}{=} |\sigma(|a|_\square)(b)|_\square$

Goldman $\rightsquigarrow$
$(\mathbb{Q}\hat{\pi}, [,])$: Lie algebra, $(\mathbb{Q}\pi_\square, [,])$: Lie algebra
Kawazumi-Kuno $\rightsquigarrow$

- $\mathbb{Q}\pi$ is $\mathbb{Q}\hat{\pi}$-module with σ.
- $\mathbb{Q}\pi$ is $\mathbb{Q}\pi_\square$-module with σ.

The definition of $\kappa : \mathbb{Q}\pi \to \ker\epsilon/(\ker\epsilon)^2$

$r \in \pi$, $L_r \in \mathcal{T}(\Sigma)$ s.t. $p_1(L_r) = |r|_\square$ ($\Rightarrow |L_r| = 1$)

$w(L_r) \overset{\text{def.}}{=} \sharp$(the set of positive crossing of L_r)

-$\sharp$(the set of negative corssing of L_r)

$\kappa(r) \overset{\text{def.}}{=} \langle L_r \rangle + 2 - 3(A - A^{-1}) w(L_r) \in \ker\epsilon/(\ker\epsilon)^2$

$\kappa(r)$ is independent on the choice of L_r.

κ induces

- $\kappa : \mathbb{Q}\pi \to \ker\epsilon/(\ker\epsilon)^2$,
- $\kappa_\square : \mathbb{Q}\pi_\square \to \ker\epsilon/(\ker\epsilon)^2$.

Remarks of κ 1/2

Remark

For x and $y \in \pi$,

$$\kappa(xy) = \kappa(yx),$$
$$\kappa(x) = \kappa(x^{-1}),$$
$$\kappa(xy) + \kappa(xy^{-1}) = 2\kappa(x) + 2\kappa(y).$$

Remark

$\kappa_\square$ is Lie algebra homomorphism. In other words, for x and $y \in \mathbb{Q}\pi_\square$, $\kappa_\square([x,y]) = [\kappa_\square(x), \kappa_\square(y)]$.

Remarks of κ 2/2

Remark

The $\mathbb{Q}$-module homomorphism

$$\begin{aligned} \mathbb{Q}(A+1)\oplus \quad \mathbb{Q}\pi/\mathbb{Q}1 &\to \ker\epsilon/(\ker\epsilon)^2 \\ A+1 \quad &\mapsto A+1 \\ r\in\pi &\mapsto \kappa(r) \end{aligned}$$

is surjective.

The augmentation map $\epsilon_\pi : \mathbb{Q}\pi \to \mathbb{Q}$ is defined by $r \in \pi \mapsto 1$.

The basis of $\ker\epsilon/(\ker\epsilon)^2$ and $\lambda : \wedge^3 H \to \ker\epsilon/(\ker\epsilon)^2$

Theorem (T.)

- $\mathbb{Q}\pi/((\ker\epsilon_\pi)^4 + \mathbb{Q}1) \to \ker\epsilon/(\ker\epsilon)^2$ *is well-defined.*
- *If* π *is a free group generated by* $d_1, d_2, \cdots, d_N$, *then* $\ker\epsilon/(\ker\epsilon)^2$ *is* $\mathbb{Q}$*-module with basis*

$$\{A+1\} \sqcup \{\langle d_i, d_j\rangle | i \leq j\} \sqcup \{\langle d_i, d_j, d_k\rangle | i < j < k\}.$$

Here $\langle d_i, d_j\rangle \stackrel{\text{def.}}{=} \kappa((d_i-1)(d_j-1))$, $\langle d_i, d_j, d_k\rangle \stackrel{\text{def.}}{=} \kappa((d_i-1)(d_j-1)(d_k-1))$.

Corollary

$$\begin{aligned} \lambda : H \wedge H \wedge H &\to \ker\epsilon/(\ker\epsilon)^2 \\ a \wedge b \wedge c &\mapsto \langle \alpha, \beta, \gamma\rangle ([\alpha] = a, [\beta] = b, [\gamma] = c \in H) \end{aligned}$$

is well-defined and injective.

Out line of proof

(generate) Use the lemma.

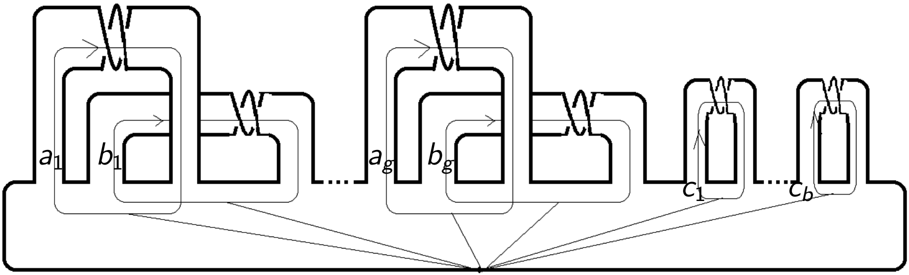
Lemma

- *For a and $r \in \pi$,*
  $\kappa(a^2 r) = -\kappa(r) + 2\kappa(ar) + \kappa(a)$.
- *For a, b, c and $d \in \pi$,*
  $\kappa((a-1)(b-1)(c-1)(d-1)) = 0$.

(linear independent)
Use Kauffman bracket.

Proof of the linear independence 1/3

Let Σ be a compact connected oriented surface with genus g and $b+1$ boundary components.
Fix an embeding $e : \Sigma \times I \to S^3$ and $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_b \in \pi_1(\Sigma)$ as the figure.
Here $2g + b = N$.



Proof of the linear independence 2/3

We fix $d_1 = a_1, d_2 = b_1, \cdots, d_{2g-1}, d_{2g}, d_{2g+1} = c_1, \cdots, d_{2g+b} = c_b$.

Define $v : \mathcal{S}(\Sigma) \to \mathbb{Q}[A, A^{-1}]$ by $\langle T \rangle \mapsto \langle e(T) \rangle$ for $T \in \mathcal{T}(\Sigma)$.

We remark that v is NOT a $\mathbb{Q}$-algebra homomorphism.

Fact

$v((\ker \epsilon)^n) \subset (A+1)^n \mathbb{Q}[A, A^{-1}]$

$Q \stackrel{\text{def.}}{=} q(A+1) + \sum_{i \leq j} q_{ij} \langle d_i, d_j \rangle + \sum_{i>j>k} q_{ijk} \langle d_i, d_j, d_k \rangle$

Proof of the linear independence 3/3

We assume $Q \in (\ker \epsilon)^2$. Denote $t \overset{\text{def.}}{=} A+1$.

For $i > j$, $v(Q\langle d_i, d_j\rangle) = 48 q_{ij} t^2 \mod t^3$

$v(Q\langle d_i, d_i\rangle) = (-6q + 240 q_{ii}) t^2 \mod t^3$

$v(Q(A+1)) = (q - 6\sum_{i=1}^{N} q_{ii}) t^2 \mod t^3$

$\rightsquigarrow q_{ij} = 0, -6q + 240 q_{ii} = 0, q - 6\sum_{i=1}^{N} q_{ii} = 0$

$\rightsquigarrow q_{ij} = 0, q_{ii} = 0, q = 0$

$v(Q^2) = 192(\sum_{i<j<K} {q_{ijk}}^2) t^3 \mod t^4$

$\rightsquigarrow \sum_{i<j<K} {q_{ijk}}^2 = 0 \rightsquigarrow q_{ijk} = 0$

Filtrations

$$\delta : \ker \epsilon \to (\ker \epsilon / (\ker \epsilon)^2)/\mathbb{Q}\mathrm{im}\lambda$$

Lemma

$(\ker \delta)^2 \subset (\ker \epsilon)^3$, $[\ker \epsilon, \ker \epsilon] \subset \ker \epsilon$
$[\ker \epsilon, \ker \delta] \subset \ker \delta$, $[\ker \delta, \ker \delta] \subset (\ker \epsilon)^2$

$$F^0\mathcal{S}(\Sigma) \stackrel{\mathrm{def.}}{=} F^1\mathcal{S}(\Sigma) \stackrel{\mathrm{def.}}{=} \mathcal{S}(\Sigma),$$
$$F^2\mathcal{S}(\Sigma) \stackrel{\mathrm{def.}}{=} \ker \epsilon, \ F^3\mathcal{S}(\Sigma) \stackrel{\mathrm{def.}}{=} \ker \delta,$$
$$F^n\mathcal{S}(\Sigma) \stackrel{\mathrm{def.}}{=} \ker \epsilon F^{n-2}\mathcal{S}(\Sigma) \text{ for } n \leq 4$$

Proposition

$[F^n\mathcal{S}(\Sigma), F^m\mathcal{S}(\Sigma)] \subset F^{m+n-2}\mathcal{S}(\Sigma)$
$F^n\mathcal{S}(\Sigma)F^m\mathcal{S}(\Sigma) \subset F^{m+n}\mathcal{S}(\Sigma)$

Completions

$$\widehat{\mathcal{S}(\Sigma)} \stackrel{\mathrm{def.}}{=} \varprojlim_{i \to \infty} \mathcal{S}(\Sigma)/F^i\mathcal{S}(\Sigma)$$

$$\widehat{\mathcal{S}(\Sigma, J)} \stackrel{\mathrm{def.}}{=} \varprojlim_{i \to \infty} \mathcal{S}(\Sigma, J)/F^i\mathcal{S}(\Sigma)\mathcal{S}(\Sigma, J)$$

Theorem (T.)

Σ: *compact connected oriented surface with non-empty boundary*

$\mathcal{S}(\Sigma) \to \widehat{\mathcal{S}(\Sigma)}$: *injective*

$\mathcal{S}(\Sigma, J) \to \widehat{\mathcal{S}(\Sigma, J)}$: *injective*

Is the quotient map injective if Σ is closed?

The logarithms of Dehn twists

Theorem (T.)

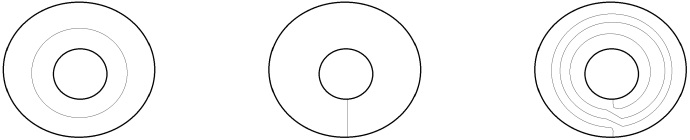
Σ: compact connected oriented surface
$J \subset \partial\Sigma$: finite subset, c: s.c.c. in Σ
$L(c) \stackrel{\text{def.}}{=} \frac{-A-A^{-1}}{4\log(-A)}(\cosh^{-1}(-\frac{c}{2}))^2 \in \widehat{\mathcal{S}(\Sigma)}$

$$\log t_c(\cdot) \stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \frac{-1}{i}(\mathrm{id} - t_c)^i(\cdot) = \sigma(L(c))(\cdot) : \widehat{\mathcal{S}(\Sigma, J)} \to \widehat{\mathcal{S}(\Sigma, J)}$$

In other words,

$$t_c(\cdot) = \exp(\sigma(L(c))) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!}(\sigma(L(c)))^i \in \mathrm{Aut}(\widehat{\mathcal{S}(\Sigma, J)})$$

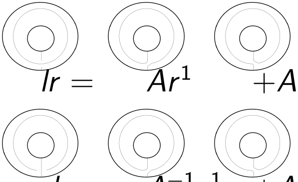
Out line of proof 1/2



l $\qquad r^0 \qquad r^{-3}$

$r^n \stackrel{\text{def.}}{=} (t_l)^n(r^0)$ for $n \in \mathbb{Z}$

Using skein relation



$lr = Ar^1 + A^{-1}r^{-1}$

$rl = A^{-1}r^1 + Ar^{-1}$

Outline of proof 2/2

$$\sigma\left(\frac{-A+A^{-1}}{4\log(-A)}(\cosh^{-1}(-\frac{l}{2}))^2\right)(r^o)$$
$$= \frac{1}{4\log(-A)}((\cosh^{-1}(-\frac{l}{2})r^0 - r^0(\cosh^{-1}(-\frac{l}{2})^2)$$
$$= \frac{1}{4\log(-A)}((\log(-Ar))^2 - (\log(-A^{-1}r))^2)$$
$$(\because (\cosh^{-1}(\frac{x+x^{-1}}{2}) = \log(x))$$
$$= \frac{1}{4\log(-A)}((\log(-A))^2 r^0 + 2\log(-A)\log(r) + (\log(r))^2$$
$$-(\log(-A))^2 r^0 + 2\log(-A)\log(r) - (\log(r))^2)$$
$$= \log(r) = \log(t_l)(r^0)$$

Goldman Lie algebra version

Theorem (Kawazumi-Kuno, Massuyeau-Turaev)

c:s.c.c. in Σ, t_c: Dehn twist along c

Then we have

$$\log(t_c) = \sigma(|\frac{1}{2}(\log(c))^2|) : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi}.$$

Here $\widehat{\mathbb{Q}\pi} \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty} \mathbb{Q}\pi/(\ker \epsilon_\pi)^i$.

Toreilli groups and Johnson homomorphisms

Σ: compact connected oriented surface with nonempty connecting boundary

$J = \{p_0, p_1\} \subset \partial\Sigma$

Remark

$\mathcal{M}(\Sigma) \curvearrowright \widehat{\mathcal{S}(\Sigma, J)}$, faithful

$\mathcal{M}(\Sigma) \hookrightarrow \mathrm{Aut}(\widehat{\mathcal{S}(\Sigma, J)})$

Baker Campbel Hasudorff series

For $x, y \in F^3\widehat{\mathcal{S}(\Sigma)}$,
$\mathrm{bch}(x, y) \stackrel{\mathrm{def.}}{=}$
$x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x[x, y]] + [y, [y, x]]) \cdots$

Fact

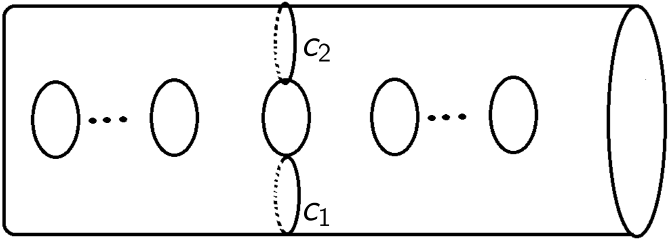
$(F^3\widehat{\mathcal{S}(\Sigma)}, \mathrm{bch})$: *group*
$\exp : (F^3\widehat{\mathcal{S}(\Sigma)}, \mathrm{bch}) \to \mathrm{Aut}(\widehat{\mathcal{S}(\Sigma, J)})$ *:group homomorphism*

Torelli group

$\mathcal{I}(\Sigma) \subset \mathcal{M}(\Sigma)$: Toreilli group
$(\xi \in \mathcal{I}(\Sigma) \Leftrightarrow \forall a \in H, \xi(a) = a)$

Fact

$\mathcal{I}(\Sigma)$ is generated by $\{t_{c_1} t_{c_2}{}^{-1} | c_1, c_2 : \mathrm{BP}\}$.



Key lemma

Lemma

c_1, c_2: BP s.t. $c_1 = |r \prod_{i=1}^{k} [a_i, b_i]|_{\square}$, $c_2 = |r|_{\square}$
for some $a_1, b_1, \cdots, a_k, b_k, r \in \pi$
Then we have
$L(c_1) - L(c_2) = \lambda(-\sum_{i=1}^{k} [r] \wedge [a_i] \wedge [b_i]) \mod F^4\widehat{\mathcal{S}(\Sigma)}$
In particular, $L(c_1) - L(c_2) \in F^3\widehat{\mathcal{S}(\Sigma)}$.

Corollary

For all $\xi \in \mathcal{I}(\Sigma)$*, there exists* $x_\xi \in F^3\widehat{\mathcal{S}(\Sigma)}$ *such that* $\xi = \exp(x_\xi)$.

Proof of the lemma 1/2

In this proof, all equations are in $\mathcal{S}(\Sigma)/(\ker\epsilon)^2 = \mathcal{S}(\Sigma)/F^4\mathcal{S}(\Sigma)$.
For $R, x, y \in \pi$, we have

$$\begin{aligned}\kappa(Rxy) &= -\kappa(Ry^{-1}x^{-1}) + 2\kappa(R) + 2\kappa(xy)\\ &= \kappa(Ry^{-1}x) - 2\kappa(Ry^{-1}) - 2\kappa(x) + 2\kappa(R) + 2\kappa(xy)\\ &= -\kappa(Ryx) + 2\kappa(Rx) + 2\kappa(y) + 2\kappa(Ry)\\ &\quad - 4\kappa(R) - 4\kappa(y) - 2\kappa(x) + 2\kappa(R) + 2\kappa(xy)\\ &= -\kappa(Ryx) + 2\kappa(Rx + Ry + xy - R - x - y).\end{aligned}$$

Hence we have

$$\kappa(Rxy - Ryx) = \kappa((R-1)(x-1)(y-a)).$$

Proof of the lemma 2/2

We have

$$
\begin{aligned}
L(c_1) - L(c_2) &= -\frac{1}{2}(c_1 c_2) \\
&= -\frac{1}{2}(\kappa(r\prod_{i=1}^{k}[a_i, b_i] - r)) \\
&= -\frac{1}{2}\sum_{j=1}^{k}\kappa(r\prod_{i=1}^{j}[a_i, b_i] - r\prod_{i=1}^{j-1}[a_i, b_i]) \\
&= -\sum_{j=1}^{k}\lambda([r\prod_{i=1}^{j-1}[a_i, b_i]a_j b_j] \wedge [a_i] \wedge [b_i]) \\
&= \lambda(-\sum_{i=1}^{k}[r] \wedge [a_i] \wedge [b_i]).
\end{aligned}
$$

Johnson homomorphism and skein algebra

Using the lemma we have the following.
$I(\widehat{\mathcal{S}(\Sigma)})$: the subgroup of $F^3\widehat{\mathcal{S}(\Sigma)}$ generated by $\{L(c_1) - L(c_2) | c_1, c_2 : \mathrm{BP}\}$.

Theorem (T.)

$\tau : \mathcal{I}(\Sigma) \to \wedge^3 H$:*(1st) Johonson homomorphism*

$$\begin{array}{ccccc} & & I(\widehat{\mathcal{S}(\Sigma)}) & \to & F^3\widehat{\mathcal{S}(\Sigma)}/F^4\widehat{\mathcal{S}(\Sigma)} \\ & \exp \swarrow \circlearrowright & \downarrow & \circlearrowright & \uparrow \lambda \\ \mathrm{Aut}(\widehat{\mathcal{S}(\Sigma, J)}) & \hookleftarrow & \mathcal{I}(\Sigma) & \underset{\tau}{\to} & \wedge^3 H \end{array}$$