Generalized torsion elements in the knot groups of twist knots

Masakazu Teragaito

Hiroshima University

Geometry and Topology of Low-dimensional Manifolds

Outline

Outline

Notation

In a group G, we use the following notation.

- $\bar{g} = g^{-1}$
- $g^a = \bar{a}ga$
- $[a, b] = \bar{a}\bar{b}ab$

Knot group

For a knot K in the 3-sphere S^3,

$$G = \pi_1(S^3 - K)$$

is called the knot group of K.

A classical fact (Papakyriakopoulos 1957)

Any knot group is torsion-free.

Knot group

For a knot K in the 3-sphere S^3,

$$G = \pi_1(S^3 - K)$$

is called the knot group of K.

A classical fact (Papakyriakopoulos 1957)

Any knot group is torsion-free.

Generalized torsion element

Let G be a group.

Definition

A non-trivial element $g \in G$ is called a generalized torsion (element) if some non-empty finite product of its conjugates equals to the identity, i.e.

$$g^{a_1} g^{a_2} \dots g^{a_n} = 1.$$

Example

For the Klein bottle S,

$$\pi_1(S) = \langle x, y \mid \bar{x} y x = \bar{y} \rangle.$$

Then $y \neq 1$ and $yy^x = 1$, so y is a generalized torsion.

Generalized torsion element

Let G be a group.

Definition

A non-trivial element $g \in G$ is called a generalized torsion (element) if some non-empty finite product of its conjugates equals to the identity, i.e.

$$g^{a_1} g^{a_2} \dots g^{a_n} = 1.$$

Example

For the Klein bottle S,

$$\pi_1(S) = \langle x, y \mid \bar{x}yx = \bar{y} \rangle.$$

Then $y \neq 1$ and $yy^x = 1$, so y is a generalized torsion.

Torus knot groups

Although any knot group is torsion-free, it may contain a generalized torsion.

Trefoil

Let K be a trefoil. Then the knot group has a presentation

$$G = \langle a, b \mid a^2 = b^3 \rangle.$$

Let $\boxed{D = [a, b]}$. Then $D \neq 1$, but

$$D^a D = (\bar{a}[a,b]a)[a,b] = \bar{a}(\bar{a}\bar{b}ab)a(\bar{a}\bar{b}ab) = \bar{a}^2\bar{b}a^2b = 1.$$

Hence D is a generalized torsion in G.

Torus knot groups

Although any knot group is torsion-free, it may contain a generalized torsion.

Trefoil

Let K be a trefoil. Then the knot group has a presentation

$$G = \langle a, b \mid a^2 = b^3 \rangle.$$

Let $\boxed{D = [a, b]}$. Then $D \neq 1$, but

$$D^a D = (\bar{a}[a,b]a)[a,b] = \bar{a}(\bar{a}\bar{b}ab)a(\bar{a}\bar{b}ab) = \bar{a}^2\bar{b}a^2b = 1.$$

Hence D is a generalized torsion in G.

Torus knot groups

In general, we have

- $[x^n, y] = [x^{n-1}, y]^x [x, y]$,
- $[x, y^n] = [x, y][x, y^{n-1}]^y$.

Thus, for $p, q > 0$,

- $[x^p, y^q]$ is a product of conjugates of $[x, y]$.
- If $[x, y] \neq 1$ and $[x^p, y^q] = 1$, then $[x, y]$ is a generalized torsion.

Torus knot groups

Any non-trivial torus knot group has a generalized torsion. This also holds for cable knots, or more generally, any knot which has either a torus knot space or a cable space in the torus decomposition of the exterior.

Torus knot groups

In general, we have

- $[x^n, y] = [x^{n-1}, y]^x [x, y]$,
- $[x, y^n] = [x, y][x, y^{n-1}]^y$.

Thus, for $p, q > 0$,

- $[x^p, y^q]$ is a product of conjugates of $[x, y]$.
- If $[x, y] \neq 1$ and $[x^p, y^q] = 1$, then $[x, y]$ is a generalized torsion.

Torus knot groups

Any non-trivial torus knot group has a generalized torsion. This also holds for cable knots, or more generally, any knot which has either a torus knot space or a cable space in the torus decomposition of the exterior.

Torus knot groups

In general, we have

- $[x^n, y] = [x^{n-1}, y]^x [x, y]$,
- $[x, y^n] = [x, y][x, y^{n-1}]^y$.

Thus, for $p, q > 0$,

- $[x^p, y^q]$ is a product of conjugates of $[x, y]$.
- If $[x, y] \neq 1$ and $[x^p, y^q] = 1$, then $[x, y]$ is a generalized torsion.

Torus knot groups

Any non-trivial torus knot group has a generalized torsion. This also holds for cable knots, or more generally, any knot which has either a torus knot space or a cable space in the torus decomposition of the exterior.

Ordering

Definition

A group G is said to be left-orderable (LO) if it admits a strict total ordering "$<$" which is invariant under left multiplication:

$$a < b \Longrightarrow ga < gb \quad (a, b, g \in G)$$

* If G is LO, then it is also right-orderable (RO).

Definition

A group G is said to be bi-orderable (BO) if it admits a strict total ordering which is invariant under right and left multiplication:

$$a < b \Longrightarrow gah < gbh \quad (a, b, g, h \in G)$$

Ordering

Definition

A group G is said to be left-orderable (LO) if it admits a strict total ordering "$<$" which is invariant under left multiplication:

$$a < b \Longrightarrow ga < gb \quad (a, b, g \in G)$$

* If G is LO, then it is also right-orderable (RO).

Definition

A group G is said to be bi-orderable (BO) if it admits a strict total ordering which is invariant under right and left multiplication:

$$a < b \Longrightarrow gah < gbh \quad (a, b, g, h \in G)$$

Ordering

Definition

A group G is said to be left-orderable (LO) if it admits a strict total ordering "$<$" which is invariant under left multiplication:

$$a < b \Longrightarrow ga < gb \quad (a, b, g \in G)$$

* If G is LO, then it is also right-orderable (RO).

Definition

A group G is said to be bi-orderable (BO) if it admits a strict total ordering which is invariant under right and left multiplication:

$$a < b \Longrightarrow gah < gbh \quad (a, b, g, h \in G)$$

Facts

A classical fact (J. Howie 1982)

Any knot group is LO.

However, it seems that the property "BO" is strong for knot groups.

Perron-Rolfsen 2003

The figure-eight knot group is BO.

They found a sufficient condition for BO (fibered) knot group.
(Later, a necessary condition was found by Clay-Rolfsen.)

Facts

A classical fact (J. Howie 1982)

Any knot group is LO.

However, it seems that the property "BO" is strong for knot groups.

Perron-Rolfsen 2003

The figure-eight knot group is BO.

They found a sufficient condition for BO (fibered) knot group.
(Later, a necessary condition was found by Clay-Rolfsen.)

Facts

A classical fact (J. Howie 1982)

Any knot group is LO.

However, it seems that the property "BO" is strong for knot groups.

Perron-Rolfsen 2003

The figure-eight knot group is BO.

They found a sufficient condition for BO (fibered) knot group.
(Later, a necessary condition was found by Clay-Rolfsen.)

BO vs Generalized torsion

Easy observation

If a group G is BO, then G has no generalized torsion.

Proof. Let $g \neq 1$. Then $g > 1$ or $g < 1$.
Since G is BO, if $g > 1$, then

$$g^a = \bar{a}ga > \bar{a} \cdot 1 \cdot a = 1 \quad \text{for any } a \in G.$$

Thus any product of conjugates of g is bigger than 1.
(If $g_1 > 1$ and $g_2 > 1$, then $g_1 g_2 > g_2 > 1$.) □

BO vs Generalized torsion

Easy observation

If a group G is BO, then G has no generalized torsion.

Proof. Let $g \neq 1$. Then $g > 1$ or $g < 1$.
Since G is BO, if $g > 1$, then

$$g^a = \bar{a} g a > \bar{a} \cdot 1 \cdot a = 1 \quad \text{for any } a \in G.$$

Thus any product of conjugates of g is bigger than 1.
(If $g_1 > 1$ and $g_2 > 1$, then $g_1 g_2 > g_2 > 1$.) □

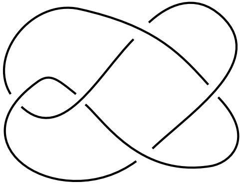
Knot table

For the first 3 prime knots in the knot table,

knot	g-torsion	BO
3_1	O	X
4_1	X	O
5_1	O	X

Naylor-Rolfsen

The next target is 5_2, which is hyperbolic.



Naylor-Rolfsen, Sep. 2014

The knot group of 5_2 has a generalized torsion.

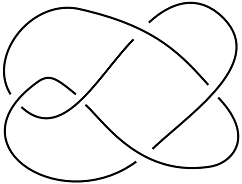
They use a presentation

$$\langle a, b \mid b^2\bar{a}^2b^2 = \bar{a}b^3\bar{a}\rangle,$$

derived from `Snappea`, and prove that $[a, \bar{b}]$ is a generalized torsion. They found it by computer.

Naylor-Rolfsen

The next target is 5_2, which is hyperbolic.



Naylor-Rolfsen, Sep. 2014

The knot group of 5_2 has a generalized torsion.

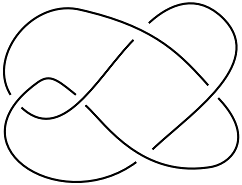
They use a presentation

$$\langle a, b \mid b^2\bar{a}^2b^2 = \bar{a}b^3\bar{a}\rangle,$$

derived from `Snappea`, and prove that $[a, \bar{b}]$ is a generalized torsion. They found it by computer.

Naylor-Rolfsen

The next target is 5_2, which is hyperbolic.



Naylor-Rolfsen, Sep. 2014

The knot group of 5_2 has a generalized torsion.

They use a presentation

$$\langle a, b \mid b^2\bar{a}^2b^2 = \bar{a}b^3\bar{a}\rangle,$$

derived from `Snappea`, and prove that $[a, \bar{b}]$ is a generalized torsion. They found it by computer.

BO=R*?

A group without generalized torsion is called an R*-group.
As said before, BO implies R*. It had been expected BO=R*, but Bludov (1972) found a counterexample.
However, it may be true for knot groups.

Conjecture

Let K be a hyperbolic knot. If the knot group G is not BO, then G has a generalized torsion.

BO=R*?

A group without generalized torsion is called an R*-group.
As said before, BO implies R*. It had been expected BO=R*, but Bludov (1972) found a counterexample.
However, it may be true for knot groups.

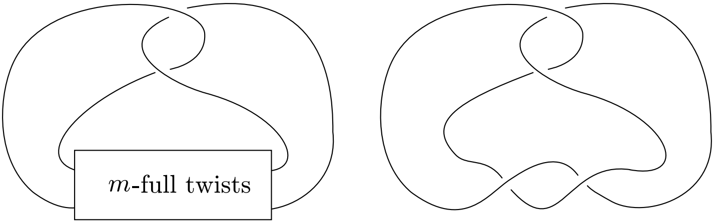
Conjecture

Let K be a hyperbolic knot. If the knot group G is not BO, then G has a generalized torsion.

Outline

Twist knots

Let K_m be the m-twist knot.

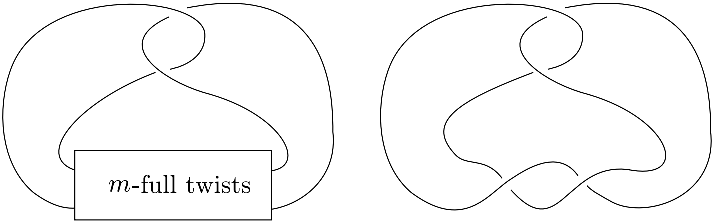


For example,

- K_1: the figure-eight knot
- K_{-1}: the right trefoil
- K_{-2}: 5_2

Twist knots

Let K_m be the m-twist knot.



For example,

- K_1: the figure-eight knot
- K_{-1}: the right trefoil
- K_{-2}: 5_2

Twist knots

Based on the result of Chiswell-Glass-Wilson (2014), Clay-Desmarais-Naylor showed:

Clay-Desmarais-Naylor, Oct. 2014

The knot group of K_m is BO if $m \geq 0$, not BO if $m < 0$.

Thus we can expect that the knot group of negative twist knot will contain a generalized torsion.

- $m = -1 \ldots 3_1$
- $m = -2 \ldots 5_2$: Naylor-Rolfsen

Twist knots

Based on the result of Chiswell-Glass-Wilson (2014), Clay-Desmarais-Naylor showed:

Clay-Desmarais-Naylor, Oct. 2014

The knot group of K_m is BO if $m \geq 0$, not BO if $m < 0$.

Thus we can expect that the knot group of negative twist knot will contain a generalized torsion.

- $m = -1 \ldots 3_1$
- $m = -2 \ldots 5_2$: Naylor-Rolfsen

Main result

Theorem

Let K_m be the m-twist knot with knot group G.

- *If $m \geq 0$, then G does not have a generalized torsion.*
- *If $m < 0$, then G has a generalized torsion.*

When $m \geq 0$, G is BO by Clay-Desmarais-Naylor. Hence G has no generalized torsion.

Main result

Theorem

Let K_m be the m-twist knot with knot group G.

- *If $m \geq 0$, then G does not have a generalized torsion.*
- *If $m < 0$, then G has a generalized torsion.*

When $m \geq 0$, G is BO by Clay-Desmarais-Naylor. Hence G has no generalized torsion.

Outline

Lin's presentation of knot group

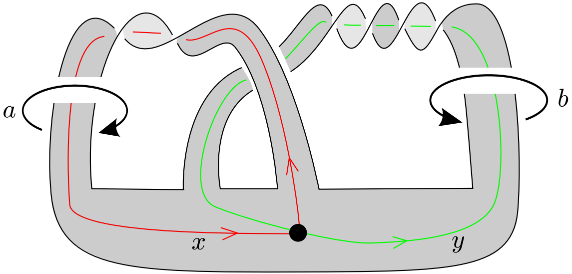
Let $n \geq 1$, and let G be the knot group of K_{-n}.

Lemma 1

G has a presentation

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

where t is a meridian.



A Seifert surface ($n = 2$)

Candidate of generalized torsion

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

From the 2nd relation, $a = [t, b^n]$.
Hence, G is generated by b and t.

Let $\boxed{D = [t, b]}$. $\longleftarrow$ This is our candidate!

Lemma 2

$D \neq 1$.

Proof. If $D = 1$, then G would be abelian. □

Candidate of generalized torsion

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

From the 2nd relation, $a = [t, b^n]$.
Hence, G is generated by b and t.

Let $\boxed{D = [t, b]}$. $\longleftarrow$ This is our candidate!

Lemma 2

$D \neq 1$.

Proof. If $D = 1$, then G would be abelian. □

Candidate of generalized torsion

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

From the 2nd relation, $a = [t, b^n]$.
Hence, G is generated by b and t.

Let $\boxed{D = [t, b]}$. $\longleftarrow$ This is our candidate!

Lemma 2

$D \neq 1$.

Proof. If $D = 1$, then G would be abelian. $\square$

The semi-group $\langle D \rangle$

Let $\langle D \rangle$ be the semi-group consisting of all (non-empty) finite products of conjugates of D.

Goal

The semi-group $\langle D \rangle$ contains the identity.

Lemma 3

$b^n, \bar{b}^n \in \langle D \rangle$.

The semi-group $\langle D \rangle$

Let $\langle D \rangle$ be the semi-group consisting of all (non-empty) finite products of conjugates of D.

Goal

The semi-group $\langle D \rangle$ contains the identity.

Lemma 3

$b^n, \bar{b}^n \in \langle D \rangle$.

The semi-group $\langle D \rangle$

Let $\langle D \rangle$ be the semi-group consisting of all (non-empty) finite products of conjugates of D.

Goal

The semi-group $\langle D \rangle$ contains the identity.

Lemma 3

$b^n, \bar{b}^n \in \langle D \rangle$.

Calculation

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

1. $(tb^n\bar{t})(t\bar{a}\bar{t}) = b^n$ (from 2nd relation)
2. $tb^n\bar{t} = b^n(ta\bar{t})$
3. $tb^n\bar{t} = b^{n-1}a$ (use 1st relation)
4. $tb^n\bar{t} = b^{n-1}[t, b^n]$ (use $a = [t, b^n]$)
5. $\bar{b}^{n-1}tb^n\bar{t} = [t, b^n]$

To show $b^n \in \langle D \rangle$, we prove

$$tb^n\bar{t} \in \langle D \rangle \longrightarrow b^{n-1}[t, b^n] \in \langle D \rangle \longrightarrow t^{n-1}(b^{n-1}[t, b^n])\bar{t}^{n-1} \in \langle D \rangle.$$

Calculation

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

1. $(tb^n\bar{t})(t\bar{a}\bar{t}) = b^n$ (from 2nd relation)
2. $tb^n\bar{t} = b^n(ta\bar{t})$
3. $tb^n\bar{t} = b^{n-1}a$ (use 1st relation)
4. $tb^n\bar{t} = b^{n-1}[t, b^n]$ (use $a = [t, b^n]$)
5. $\bar{b}^{n-1}tb^n\bar{t} = [t, b^n]$

To show $b^n \in \langle D \rangle$, we prove

$$tb^n\bar{t} \in \langle D \rangle \longrightarrow b^{n-1}[t, b^n] \in \langle D \rangle \longrightarrow t^{n-1}(b^{n-1}[t, b^n])\bar{t}^{n-1} \in \langle D \rangle.$$

Calculation for b^n

4. $tb^n\overline{t} = b^{n-1}[t, b^n]$
5. $\overline{b}^{n-1}tb^n\overline{t} = [t, b^n]$

We prove $t^{n-1}(b^{n-1}[t, b^n])\overline{t}^{n-1} \in \langle D \rangle$.

$$\begin{aligned}
t^{n-1}(b^{n-1}[t, b^n])\overline{t}^{n-1} &= t^{n-1}b^{n-1}(\overline{t}\,\overline{b}) \cdot \overline{b}^{n-1}tb^n\overline{t} \cdot \overline{t}^{n-2} \\
&= t^{n-1}b^{n-1}(\overline{t}\,\overline{b}) \cdot [t, b^n] \cdot \overline{t}^{n-2} \\
&= t^{n-1}b^{n-1}(\overline{t}\,\overline{b})^2 \cdot \overline{b}^{n-1}tb^n\overline{t} \cdot \overline{t}^{n-3} \\
&= t^{n-1}b^{n-1}(\overline{t}\,\overline{b})^2 \cdot [t, b^n] \cdot \overline{t}^{n-3} \\
&\vdots \\
&= t^{n-1}b^{n-1}(\overline{t}\,\overline{b})^{n-1} \cdot [t, b^n]
\end{aligned}$$

Note $[t, b^n] \in \langle D \rangle$. Hence, if $n = 1$, then we are done.

Calculation for b^n

4. $tb^n\bar{t} = b^{n-1}[t, b^n]$
5. $\bar{b}^{n-1}tb^n\bar{t} = [t, b^n]$

We prove $t^{n-1}(b^{n-1}[t,b^n])\bar{t}^{n-1} \in \langle D\rangle$.

$$
\begin{aligned}
t^{n-1}(b^{n-1}[t,b^n])\bar{t}^{n-1} &= t^{n-1}b^{n-1}(\bar{t}\,\bar{b}) \cdot \bar{b}^{n-1}tb^n\bar{t} \cdot \bar{t}^{n-2} \\
&= t^{n-1}b^{n-1}(\bar{t}\,\bar{b}) \cdot [t,b^n] \cdot \bar{t}^{n-2} \\
&= t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^2 \cdot \bar{b}^{n-1}tb^n\bar{t} \cdot \bar{t}^{n-3} \\
&= t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^2 \cdot [t,b^n] \cdot \bar{t}^{n-3} \\
&\ \vdots \\
&= t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} \cdot [t,b^n]
\end{aligned}
$$

Note $[t, b^n] \in \langle D\rangle$. Hence, if $n = 1$, then we are done.

Case $n \geq 2$

Goal

$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} \in \langle D \rangle.$

Claim 1

$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} = D^{(bt)^{n-2}\bar{b}^{n-1}\bar{t}^{n-1}} \cdot t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t}.$

Proof. Just calculate. This equation holds in the free group $F(t, b)$.

If $n = 2$, then the right-hand side is $D^{\bar{b}\,\bar{t}} \in \langle D \rangle$, done.

Case $n \geq 2$

Goal

$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} \in \langle D \rangle.$

Claim 1

$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} = D^{(bt)^{n-2}\bar{b}^{n-1}\bar{t}^{n-1}} \cdot t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t}.$

Proof. Just calculate. This equation holds in the free group $F(t, b)$.

If $n = 2$, then the right-hand side is $D^{\bar{b}\,\bar{t}} \in \langle D \rangle$, done.

Case $n \geq 2$

Goal

$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} \in \langle D \rangle$.

Claim 1

$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} = D^{(bt)^{n-2}\bar{b}^{n-1}\bar{t}^{n-1}} \cdot t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t}$.

Proof. Just calculate. This equation holds in the free group $F(t, b)$.

If $n = 2$, then the right-hand side is $D^{\bar{b}\,\bar{t}} \in \langle D \rangle$, done.

Case $n \geq 3$

$$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} = D^{(bt)^{n-2}\bar{b}^{n-1}\bar{t}^{n-1}} \cdot t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t}.$$

Claim 2

$$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t} = [\bar{t}, \bar{b}^{n-1}]^{\bar{t}^{n-2}} \dots [\bar{t}, \bar{b}^3]^{\bar{t}^2}[\bar{t}, \bar{b}^2]^{\bar{t}}$$

Proof. Induction on n. This equation also holds in the free group $F(t, b)$.

Since $[\bar{t}, \bar{b}^m] = [t, b^m]^{\bar{t}\,\bar{b}^m} \in \langle D \rangle$, we are done.

Case $n \geq 3$

$$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} = D^{(bt)^{n-2}\bar{b}^{n-1}\bar{t}^{n-1}} \cdot t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t}.$$

Claim 2

$$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t} = [\bar{t},\bar{b}^{n-1}]^{\bar{t}^{n-2}} \dots [\bar{t},\bar{b}^{3}]^{\bar{t}^{2}}[\bar{t},\bar{b}^{2}]^{\bar{t}}$$

Proof. Induction on n. This equation also holds in the free group $F(t,b)$.

Since $[\bar{t},\bar{b}^m] = [t,b^m]^{\bar{t}\,\bar{b}^m} \in \langle D\rangle$, we are done.

Case $n \geq 3$

$$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-1} = D^{(bt)^{n-2}\bar{b}^{n-1}\bar{t}^{n-1}} \cdot t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t}.$$

Claim 2

$$t^{n-1}b^{n-1}(\bar{t}\,\bar{b})^{n-2}\bar{b}\bar{t} = [\bar{t},\bar{b}^{n-1}]^{\bar{t}^{n-2}} \dots [\bar{t},\bar{b}^{3}]^{\bar{t}^{2}}[\bar{t},\bar{b}^{2}]^{\bar{t}}$$

Proof. Induction on n. This equation also holds in the free group $F(t,b)$.

Since $[\bar{t},\bar{b}^m] = [t,b^m]^{\bar{t}\,\bar{b}^m} \in \langle D \rangle$, we are done.

Calculation for $\bar{b}^n$

4. $tb^n\bar{t} = b^{n-1}[t, b^n]$
5. $\bar{b}^{n-1}tb^n\bar{t} = [t, b^n]$

$$tb^n\bar{t} = b^{n-1}[t, b^n] \longrightarrow tb^n\bar{t} = b^{n-1}\bar{t}\,\bar{b}^n t b^n \longrightarrow \bar{b}^{n-1}tb^n\bar{t}\,\bar{b}^n = \bar{t}\,\bar{b}^n t$$

4. $\bar{t}\,\bar{b}^n t = \bar{b}^{n-1}[\bar{t}, \bar{b}^n]$
5. $b^{n-1}\bar{t}\,\bar{b}^n t = [\bar{t}, \bar{b}^n]$

Thus we can conclude $\bar{b}^n \in \langle D \rangle$.

Calculation for $\bar{b}^n$

4. $tb^n\bar{t} = b^{n-1}[t, b^n]$
5. $\bar{b}^{n-1}tb^n\bar{t} = [t, b^n]$

$$tb^n\bar{t} = b^{n-1}[t, b^n] \longrightarrow tb^n\bar{t} = b^{n-1}\bar{t}\,\bar{b}^n t b^n \longrightarrow \bar{b}^{n-1}tb^n\bar{t}\,\bar{b}^n = \bar{t}\,\bar{b}^n t$$

4. $\bar{t}\,\bar{b}^n t = \bar{b}^{n-1}[\bar{t}, \bar{b}^n]$
5. $b^{n-1}\bar{t}\,\bar{b}^n t = [\bar{t}, \bar{b}^n]$

Thus we can conclude $\bar{b}^n \in \langle D \rangle$.

Calculation for $\bar{b}^n$

4. $tb^n\bar{t} = b^{n-1}[t, b^n]$
5. $\bar{b}^{n-1}tb^n\bar{t} = [t, b^n]$

$$tb^n\bar{t} = b^{n-1}[t, b^n] \longrightarrow tb^n\bar{t} = b^{n-1}\bar{t}\,\bar{b}^n t b^n \longrightarrow \bar{b}^{n-1}tb^n\bar{t}\,\bar{b}^n = \bar{t}\,\bar{b}^n t$$

4. $\bar{t}\,\bar{b}^n t = \bar{b}^{n-1}[\bar{t}, \bar{b}^n]$
5. $b^{n-1}\bar{t}\,\bar{b}^n t = [\bar{t}, \bar{b}^n]$

Thus we can conclude $\bar{b}^n \in \langle D \rangle$.

Look back

$n = 2$

The generalized torsion found by Naylor-Rolfsen is $[\bar{b}, t] = D^{\bar{b}}$ in our notation.

Mystery

What is the geometric meaning of D?

Knot group

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

From relations, $a = [t, b^n]$ and $b = [\bar{a}, \bar{t}]$. Hence, a, b and D survive in the lower central series of G.

Look back

$n = 2$

The generalized torsion found by Naylor-Rolfsen is $[\bar{b}, t] = D^{\bar{b}}$ in our notation.

Mystery

What is the geometric meaning of D?

Knot group

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

From relations, $a = [t, b^n]$ and $b = [\bar{a}, \bar{t}]$. Hence, a, b and D survive in the lower central series of G.

Look back

$n = 2$

The generalized torsion found by Naylor-Rolfsen is $[\bar{b}, t] = D^{\bar{b}}$ in our notation.

Mystery

What is the geometric meaning of D?

Knot group

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n \rangle,$$

From relations, $a = [t, b^n]$ and $b = [\bar{a}, \bar{t}]$. Hence, a, b and D survive in the lower central series of G.

Outline

Dehn surgery

Problem

- Which 3-manifold group is BO?
- Which 3-manifold group has a generalized torsion?

We prove:

Theorem

Let K be a negative twist knot, and let $r \in \mathbb{Q}$. Then $\pi_1 K(r)$ has a generalized torsion. Hence $\pi_1 K(r)$ is not BO.

Dehn surgery

Problem

- Which 3-manifold group is BO?
- Which 3-manifold group has a generalized torsion?

We prove:

Theorem

Let K be a negative twist knot, and let $r \in \mathbb{Q}$. Then $\pi_1 K(r)$ has a generalized torsion. Hence $\pi_1 K(r)$ is not BO.

Presentation

Let $K = K_{-n}$ be the $(-n)$-twist knot, $n \geq 1$.
Let $r = p/q \in \mathbb{Q}$ and $G = \pi_1 K(r)$.

Lemma 1

G has a presentation

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1 \rangle,$$

where t is the meridian, and $[\bar{a}, \bar{b}^n]$ is the longitude.

Presentation

Let $K = K_{-n}$ be the $(-n)$-twist knot, $n \geq 1$.
Let $r = p/q \in \mathbb{Q}$ and $G = \pi_1 K(r)$.

Lemma 1

G has a presentation

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1 \rangle,$$

where t is the meridian, and $[\bar{a}, \bar{b}^n]$ is the longitude.

Candidate of generalized torsion

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1 \rangle,$$

Let $\boxed{D = [t, b]}$. $\longleftarrow$ D was a generalized torsion before surgery!

Goal

D is still a generalized torsion in G.

Candidate of generalized torsion

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1\rangle,$$

Let $\boxed{D = [t, b]}$. $\longleftarrow$ D was a generalized torsion before surgery!

Goal

D is still a generalized torsion in G.

Candidate of generalized torsion

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1 \rangle,$$

Let $\boxed{D = [t, b]}$. $\longleftarrow$ D was a generalized torsion before surgery!

Goal

D is still a generalized torsion in G.

Proof

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1\rangle,$$

$$\boxed{D = [t, b]}$$

It suffices to show $D \neq 1$ in G.

Lemma 2

$D \neq 1$ in G.

Proof. Recall $a = [t, b^n]$. If $D = 1$, then $a = 1$. Then

$$G = \langle b, t \mid b = 1, t^p = 1\rangle = \langle t \mid t^p = 1\rangle.$$

Proof

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n \bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1 \rangle,$$

$$\boxed{D = [t, b]}$$

It suffices to show $D \neq 1$ in G.

Lemma 2

$D \neq 1$ in G.

Proof. Recall $a = [t, b^n]$. If $D = 1$, then $a = 1$. Then

$$G = \langle b, t \mid b = 1, t^p = 1 \rangle = \langle t \mid t^p = 1 \rangle.$$

Proof

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(b^n\bar{a})\bar{t} = b^n, t^p[\bar{a}, \bar{b}^n]^q = 1\rangle,$$

$$\boxed{D = [t, b]}$$

It suffices to show $D \neq 1$ in G.

Lemma 2

$D \neq 1$ in G.

Proof. Recall $a = [t, b^n]$. If $D = 1$, then $a = 1$. Then

$$G = \langle b, t \mid b = 1, t^p = 1\rangle = \langle t \mid t^p = 1\rangle.$$

Case $p = 0$

Now, $G \cong \mathbb{Z}$.
Since $K(0)$ is prime, $K(0) = S^2 \times S^1$.
This contradicts Property R.
(Only the unknot can yield $S^2 \times S^1$ by 0-surgery.)

Case $p \neq 0$

$G \cong \mathbb{Z}_p$.

- Except the case $n = 1$ (trefoil), K has no cyclic surgery.
- Let $n = 1$.
  - If $r \neq 6 + \frac{1}{m}$, then G is not cyclic.
  - If $r = 6 + \frac{1}{m}$, then $K(r)$ is a lens space. So, G has torsion.

Positive twist knots

Question

Let K be a positive twist knot. When $\pi_1 K(r)$ has a generalized torsion?

For the figure-eight knot, $\pi_1 K(0)$ is BO, so has no generalized torsion. (Perron-Rolfsen)

In general, $K(r)$, $r = 1, 2, 3$, is a small Seifert fibered manifold for any non-trivial twist knot K. Then $\pi_1 K(r)$ has a generalized torsion.

This also holds for $r = 4$, because $K(4)$ has a Seifert piece with exceptional fiber.

Positive twist knots

Question

Let K be a positive twist knot. When $\pi_1 K(r)$ has a generalized torsion?

For the figure-eight knot, $\pi_1 K(0)$ is BO, so has no generalized torsion. (Perron-Rolfsen)

In general, $K(r)$, $r = 1, 2, 3$, is a small Seifert fibered manifold for any non-trivial twist knot K. Then $\pi_1 K(r)$ has a generalized torsion. This also holds for $r = 4$, because $K(4)$ has a Seifert piece with exceptional fiber.

Positive twist knots

Question

Let K be a positive twist knot. When $\pi_1 K(r)$ has a generalized torsion?

For the figure-eight knot, $\pi_1 K(0)$ is BO, so has no generalized torsion. (Perron-Rolfsen)

In general, $K(r)$, $r = 1, 2, 3$, is a small Seifert fibered manifold for any non-trivial twist knot K. Then $\pi_1 K(r)$ has a generalized torsion.

This also holds for $r = 4$, because $K(4)$ has a Seifert piece with exceptional fiber.

Positive twist knots

Question

Let K be a positive twist knot. When $\pi_1 K(r)$ has a generalized torsion?

For the figure-eight knot, $\pi_1 K(0)$ is BO, so has no generalized torsion. (Perron-Rolfsen)
In general, $K(r)$, $r = 1, 2, 3$, is a small Seifert fibered manifold for any non-trivial twist knot K. Then $\pi_1 K(r)$ has a generalized torsion.
This also holds for $r = 4$, because $K(4)$ has a Seifert piece with exceptional fiber.

4-surgery on positive twist knot

Let $K = K_n$ $(n \geq 1)$, and $G = \pi_1 K(4)$.

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(\bar{b}^n\bar{a})\bar{t} = \bar{b}^n, t^4[\bar{a}, b^n] = 1 \rangle.$$

From 2nd relation,

$$\bar{b}^n\bar{a}\bar{t} = \bar{t}\,\bar{b}^n, \quad b^n t = tab^n.$$

Let $\boxed{D = t^2 a}$.

(Since $D \to 2$ under abelianization of G, $D \neq 1$ in G.)

$$\begin{aligned} DD^{b^n t} &= (t^2a)(\bar{t}\,\bar{b}^n)(t^2a)(b^nt) \\ &= (t^2a)(\bar{b}^n\bar{a}\bar{t})(t^2a)(b^nt) \\ &= t^2a\bar{b}^n\bar{a}tab^nt \\ &= t^2a\bar{b}^n\bar{a}b^nt^2 \\ &= (t^4[\bar{a}, b^n])^{t^2} = 1. \end{aligned}$$

4-surgery on positive twist knot

Let $K = K_n$ $(n \geq 1)$, and $G = \pi_1 K(4)$.

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(\bar{b}^n\bar{a})\bar{t} = \bar{b}^n, t^4[\bar{a}, b^n] = 1\rangle.$$

From 2nd relation,

$$\bar{b}^n\bar{a}\bar{t} = \bar{t}\bar{b}^n, \quad b^n t = tab^n.$$

Let $\boxed{D = t^2 a}$.

(Since $D \to 2$ under abelianization of G, $D \neq 1$ in G.)

$$\begin{aligned} DD^{b^n t} &= (t^2a)(\bar{t}\bar{b}^n)(t^2a)(b^n t) \\ &= (t^2a)(\bar{b}^n\bar{a}\bar{t})(t^2a)(b^n t) \\ &= t^2a\bar{b}^n\bar{a}tab^n t \\ &= t^2a\bar{b}^n\bar{a}b^n t^2 \\ &= (t^4[\bar{a}, b^n])^{t^2} = 1. \end{aligned}$$

4-surgery on positive twist knot

Let $K = K_n$ $(n \geq 1)$, and $G = \pi_1 K(4)$.

$$G = \langle a, b, t \mid ta\bar{t} = \bar{b}a, t(\bar{b}^n\bar{a})\bar{t} = \bar{b}^n, t^4[\bar{a}, b^n] = 1\rangle.$$

From 2nd relation,

$$\bar{b}^n\bar{a}\bar{t} = \bar{t}\,\bar{b}^n, \quad b^n t = tab^n.$$

Let $\boxed{D = t^2 a}$.

(Since $D \to 2$ under abelianization of G, $D \neq 1$ in G.)

$$\begin{aligned} DD^{b^n t} &= (t^2a)(\bar{t}\,\bar{b}^n)(t^2a)(b^n t) \\ &= (t^2a)(\bar{b}^n\bar{a}\bar{t})(t^2a)(b^n t) \\ &= t^2a\bar{b}^n\bar{a}tab^n t \\ &= t^2a\bar{b}^n\bar{a}b^n t^2 \\ &= (t^4[\bar{a}, b^n])^{t^2} = 1. \end{aligned}$$

References

A. Clay, C. Desmarais and P. Naylor,
Testing bi-orderability of knot groups, preprint.

G. Naylor and D. Rolfsen,
Generalized torsion in knot group, to appear in Canad. Math. Bull.

M. Teragaito,
Generalized torsion elements in the knot groups of twist knots,
to appear in Proc. Amer. Math. Soc.