An extension of the LMO functor and Milnor invariants

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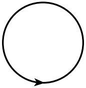
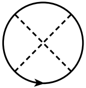
Topology and Geometry of Low-dimensional Manifolds

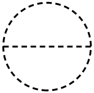
Contents

History of "LMO"

1. Kontsevich [Kon93] constructed the Kontsevich invariant Z^{K} of (unframed) oriented links in S^3, which was extended to an invariant of (framed) q-tangles.

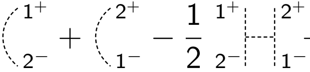
$Z^{\mathrm{K}}(\text{unknot}) =$  $- \frac{1}{24}$  $+ (\deg \geq 3) \in \mathcal{A}(\circlearrowleft)/\mathrm{FI}.$

2. Using Z^{K}, Le, Murakami and Ohtsuki [LMO98] introduced the LMO invariant of connected, oriented, closed 3-manifolds.

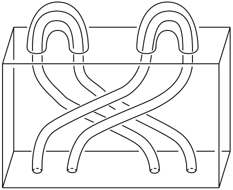
$Z^{\mathrm{LMO}}(L(p,1)) = \varnothing - \frac{p}{2}\frac{(p-1)(p-2)}{24p}$  $+ \cdots \in \mathcal{A}(\emptyset).$

3. Cheptea, Habiro and Massuyeau [CHM08] constructed the LMO functor (defined on a certain category of cobordisms) by using formal Gaussian integrals.

$$\log_{\sqcup} \widetilde{Z}(\psi_{\bullet,\bullet}) =$$



$$+ (\text{i-deg} > 2),$$

where $\psi_{\bullet,\bullet} :=$  .

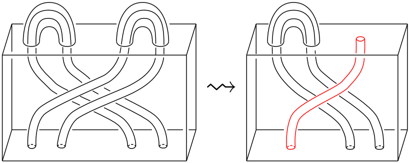
M: a $\mathbb{Q}$HS $\rightsquigarrow$ $M \setminus \mathrm{Int}[-1,1]^3$ is regarded as a cobordism between disks $[-1,1]^2 \times 1$ and $[-1,1]^2 \times (-1)$. Then

$$Z^{\mathrm{LMO}}(M) = \widetilde{Z}(M \setminus \mathrm{Int}[-1,1]^3).$$

What kind of extension?

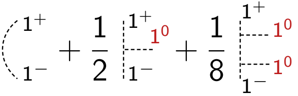
Roughly speaking, the objects and morphisms of the domain are extended as follows:

$$\Sigma_{g,1} \rightsquigarrow \Sigma_{g,1+n} \ (n \geq 0) \quad \text{and}$$



Namely, we allow a cobordim to have vertical tubes.

$$\log_{\sqcup} \widetilde{Z}(\psi_{\bullet,\circ}) =$$



$$+ \text{(i-deg} > 2).$$

Remarks

Convention

Notation and terminology are almost the same as in [CHM08], but their definitions are extended. The main differences will be emphasized in red (e.g. $\Sigma_{g,1+n}$).

Remark

Related researches are found in [ABMP10], [Kat14]. (References are listed at the end.)

More precisely, the LMO functor is defined as a tensor-preserving functor

$$\widetilde{Z}\colon \mathcal{LC}ob_q \to {}^{ts}\mathcal{A}$$

between the monoidal categories [CHM08]. $\widetilde{Z}$ has important properties:

- $\widetilde{Z}(M) = \exp_{\sqcup}(\mathrm{Lk}(M)/2) \sqcup \widetilde{Z}^Y(M)$ for $\forall M = (M, \sigma, m)$.
- $\widetilde{Z}$ is universal among rational-valued finite-type invariants of certain 3-manifolds.
- $\widetilde{Z}$ is related with Milnor invariants of string links.

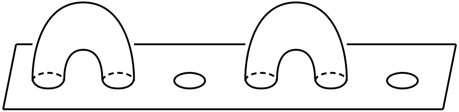
Aim of today's talk

Construct an extension of $\widetilde{Z}$ with the above properties.

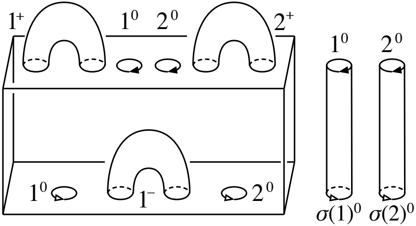
Contents

Notation

- $\mathrm{Mon}(\bullet, \circ) :=$ (free monoid generated by letters $\bullet$ and $\circ$).
- $w \in \mathrm{Mon}(\bullet, \circ)$. The compact surface F_w is defined as follows:

$$F_{\bullet\circ\bullet\circ} =$$  $$\subset \mathbb{R}^3.$$

- $w_\pm \in \mathrm{Mon}(\bullet, \circ)$ ($|w_+^\circ| = |w_-^\circ| =: n$), $\sigma \in \mathfrak{S}_n$.
  The closed surface $R^{w_+}_{w_-,\sigma}$ is defined as follows:

$$R^{\bullet\circ\circ\bullet}_{\circ\bullet\circ,\sigma} =$$ 

$$\Big/ 1^0, 2^0.$$

Domain of $\widetilde{Z}$

Definition (cobordism)

A *cobordism* from F_{w_+} to F_{w_-} ($|w_+^\circ| = |w_-^\circ| =: n$) is an equivalence class of triples (M, σ, m), where

- M is a connected, oriented, compact 3-manifold s.t. $\partial M \cong \Sigma_{|w_+^\bullet|+|w_-^\bullet|+n}$,
- $\sigma \in \mathfrak{S}_n$,
- $m\colon R^{w_+}_{w_-,\sigma} \to \partial M$ is an orientation-preserving homeomorphism,
- $(M, \sigma, m) \sim (N, \tau, n)$ if $\sigma = \tau$ and there is an ori.-pres. homeo. $f\colon M \to N$ s.t.

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {}^{m}\nwarrow \quad \circlearrowleft \quad \nearrow^{n} & \\ & R^{w_+}_{w_-,\sigma} & \end{array}.$$

Definition (strict monoidal category $\mathcal{C}ob$)

- $\mathrm{Obj}(\mathcal{C}ob) := \mathrm{Mon}(\bullet, \circ)$.
- $\mathcal{C}ob(w_+, w_-) := \{\text{cobordisms from } F_{w_+} \text{ to } F_{w_-}\}$ (or $\emptyset$).
- $(M, \sigma, m) \circ (N, \tau, n) := (M \cup_{m_+ \circ n_-^{-1}} N, \sigma\tau, m_- \cup n_+)$.
- $\mathrm{Id}_w := (F_w \times [-1, 1], \mathrm{Id}_{\mathfrak{S}_{|w^\circ|}}, \text{“Id”})$.
- $(M, \sigma, m) \otimes (N, \tau, n) :=$ (horizontal juxtaposition of M and N).

$m_\pm$ is the restriction of m to the top/bottom of the surface $R^{w_+}_{w_-,\sigma}$.

For a technical reason, we only consider *Lagrangian* cobordisms that satisfy almost the same homological conditions as in [CHM08].
$\rightsquigarrow$ The strict monoidal subcategory $\mathcal{LC}ob$.

Bottom-top tangles

Translating cobordisms into “bottom-top tangles” is necessary to define $\widetilde{Z}$. In fact, there is a 1-1 correspondence by digging a bottom-top tangle (B, γ) along its framed oriented tangle γ.

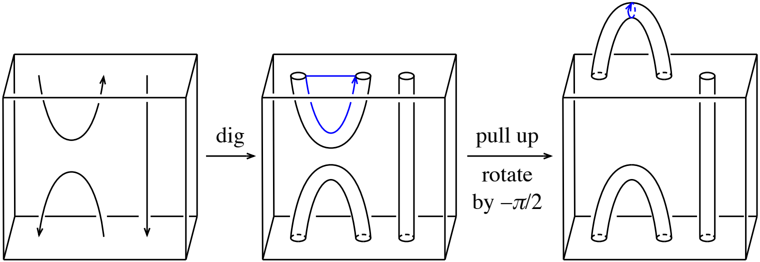


Figure : $(B, \gamma) \overset{1:1}{\longleftrightarrow} (M, \sigma, m) \in \mathcal{C}ob(\bullet\circ, \bullet\circ)$

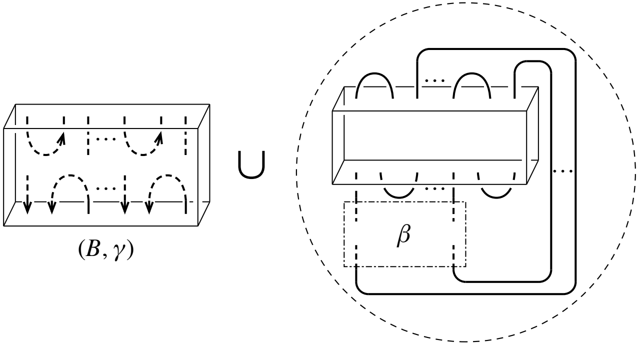
Under the previous correspondence, (M, σ, m) is Lagrangian iff $H_*(B) \cong H_*([-1,1]^3)$ & $\mathrm{Lk}_B(\gamma^+) = O$. Here,

$$\mathrm{Lk}_B(\gamma) := \mathrm{Lk}_{\widehat{B}}(\widehat{\gamma}) - O_{g_+ + g_-} \oplus \sigma^{-1} \cdot \mathrm{Cr}(\beta) \in \tfrac{1}{2}\,\mathrm{Sym}_{\pi_0\gamma}(\mathbb{Z}),$$

where $\mathrm{Lk}_{\widehat{B}}(\widehat{\gamma})$ is the usual linking matrix of the (framed) link

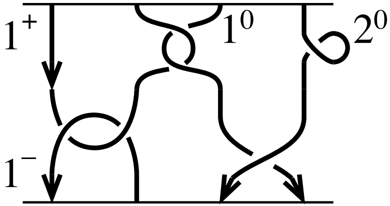
$$\widehat{\gamma} := \gamma \cup (\text{arcs and braid in } S^3 \setminus \mathrm{Int}[-1,1]^3)$$

in the homology sphere $\widehat{B} := B \cup (S^3 \setminus \mathrm{Int}[-1,1]^3)$.



In the case of $B = [-1, 1]^3$, it suffices to count the crossings of a projection of γ.

Example $((B, \gamma) = ([-1, 1]^3, \text{figure below}))$



$$\begin{array}{cccc} 1^+ & 1^- & 1^0 & 2^0 \end{array}$$
$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & -1 \end{pmatrix}$$

Moreover, (the corresponding cobordism of) (B, γ) is Lagrangian since $\mathrm{Lk}_B(\gamma^+) = (0)$.

Codomain of $\widetilde{Z}$

Definition (space of Jacobi diagrams)

X: an oriented compact 1-manifold, C: a finite set.
$\mathcal{A}(X, C) := \mathbb{Q}\{\text{Jacobi diagrams based on } (X, C)\}/\text{AS, IHX, STU}$.

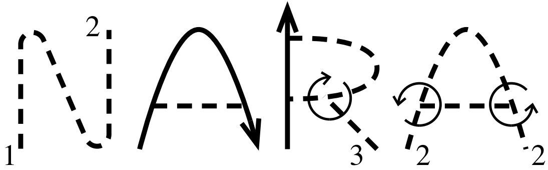


Figure : $X = \frown\!\!\searrow\uparrow$, $C = \{1, 2, 3\}$, $\deg = 12/2 = 6$

Remark

Take the degree completion of $\mathcal{A}(X, C)$ and denote it by $\mathcal{A}(X, C)$ again.

The graded $\mathbb{Q}$-linear map $\chi_S \colon \mathcal{A}(X, C \cup S) \to \mathcal{A}(X \downarrow^S, C)$ is defined as follows.

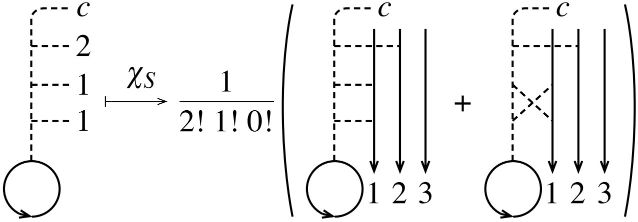


Figure : $X = \circlearrowleft$, $c \in C$, $S = \{1, 2, 3\}$

χ_S is an isomorphism and plays an important role when dealing with Jacobi diagrams.
Moreover, we need a graded $\mathbb{Q}$-linear map $\chi_{S,S'} \colon \mathcal{A}(X, C \cup S \cup S') \to \mathcal{A}(X \downarrow^S, C)$, where S' is a copy of S.

Definition (strict monoidal category ${}^{ts}\mathcal{A}$)

- $\mathrm{Obj}({}^{ts}\mathcal{A}) := \mathbb{Z}^2_{\geq 0}$.
- ${}^{ts}\mathcal{A}((g,n),(f,n)) := \{x \in \mathcal{A}(\emptyset, \lfloor g \rceil^+ \cup \lfloor f \rceil^- \cup \lfloor n \rceil^0) \mid$ x is a series of top-substantial Jacobi diagrams$\}$.
- $x \circ y := \chi^{-1}_{\lfloor n \rceil^0} \chi_{\lfloor n \rceil^0, \lfloor n \rceil^{0'}} \left\langle (x/i^+ \mapsto i^*), (y/i^- \mapsto i^*, i^0 \mapsto i^{0'}) \right\rangle_{\lfloor g \rceil^*}$.
- $\mathrm{Id}_{(g,n)} := \exp_{\sqcup} \left(\sum_{i=1}^{g}$ 

$\right)$.
- $x \otimes y := x \sqcup y$.

$\lfloor k \rceil^* := \{1^*, 2^*, \ldots, k^*\}$.

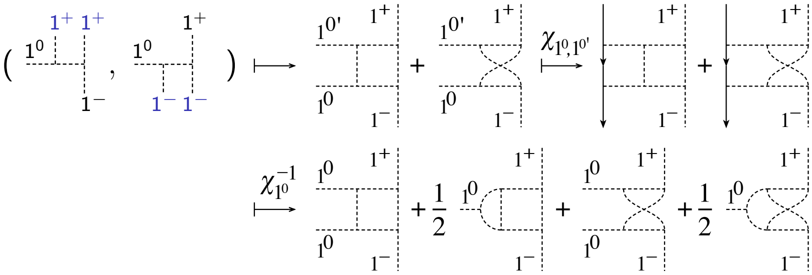
A Jacobi diagram is *top-substantial* if it contains *no* strut 

.

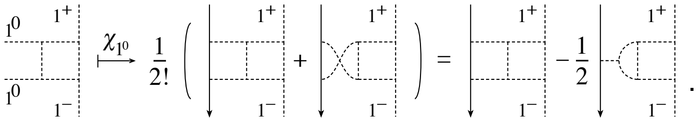
$$(x,y) \in {}^{ts}\mathcal{A}((g,n),(f,n)) \times {}^{ts}\mathcal{A}((h,n),(g,n)) \xrightarrow{\langle -,- \rangle_{\lfloor g \rceil^*}}$$

$$\mathcal{A}(\emptyset, h^+ \cup f^- \cup n^0 \cup n^{0'}) \xrightarrow{\chi_{n^0, n^{0'}}} \mathcal{A}(\downarrow^{n^0}, h^+ \cup f^-) \xrightarrow{\chi^{-1}_{n^0}} \mathcal{A}(\emptyset, h^+ \cup f^- \cup n^0).$$

Example ($f = g = n = 1$)



where the last step follows from



Before stating the main result...

Remark

The Kontsevich invariant of tangles depends on a "parenthesizing" of their boundaries. (In other words, it depends on the choice of an "associator".) Therefore, we have to refine as follows.

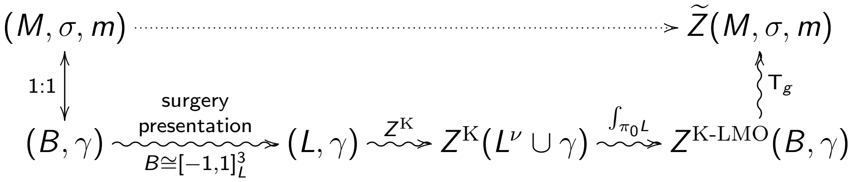
until now	from now on
$\mathrm{Mon}(\bullet, \circ)$	$\mathrm{Mag}(\bullet, \circ)$
$\mathcal{C}ob$	$\mathcal{C}ob_q$
$\mathcal{LC}ob$	$\mathcal{LC}ob_q$

There is a canonical surjection $\mathrm{Mon}(\bullet, \circ) \xleftarrow{\text{forget}} \mathrm{Mag}(\bullet, \circ)$, e.g., $\bullet \circ \circ \bullet \leftarrow\!\shortmid \bullet(\circ(\circ\bullet))$.

Let $(M, \sigma, m) \in \mathcal{LC}ob_q(w_+, w_-)$ and $g = |w_+^\bullet|$, $f = |w_-^\bullet|$, $n = |w_+^\circ|$.

Definition (extension of the LMO functor)

$$\widetilde{Z}(M, \sigma, m) := \chi_{\pi_0\gamma}^{-1} Z^{\text{K-LMO}}(B, \gamma) \circ_{^{ts}\!\mathcal{A}} \mathsf{T}_g \in {}^{ts}\!\mathcal{A}((g, n), (f, n)).$$



- $\int_{\pi_0 L}$ is the formal Gaussian integral along $\pi_0 L$ introduced in [BNGRT02a, BNGRT02b, BNGRT04].
- $Z^{\text{K-LMO}}(B, \gamma) \in \mathcal{A}(\downarrow^\gamma, \emptyset) \cong \mathcal{A}(\emptyset, \pi_0\gamma)$.
- $\mathsf{T}_g \in \mathcal{A}(\emptyset, \lfloor g \rceil^+ \cup \lfloor g \rceil^-)$ is the same as defined in [CHM08].

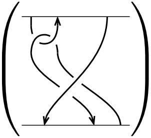
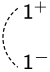
Theorem (N. '15)

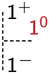
$\widetilde{Z}\colon \mathcal{LC}ob_q \to {}^{ts}\mathcal{A}$ *is a tensor-preserving functor that splits as*

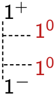
$$\widetilde{Z}(M,\sigma,m) = \exp_\sqcup(\mathrm{Lk}_B(\gamma)/2) \sqcup \widetilde{Z}^Y(M,\sigma,m).$$

Advantage

The above $\widetilde{Z}$ is a non-trivial extension of the LMO functor, indeed, $\widetilde{Z}$ reflects interaction between the top/bottom components and the vertical components:

$\log_\sqcup \widetilde{Z}$  $=$ 

$+ \frac{1}{2}$ 

$+ \frac{1}{8}$ 

$+$ (i-deg > 2).

Contents

String links

A *string link* (B, σ) on ℓ strands is, for example, the figure in the middle.

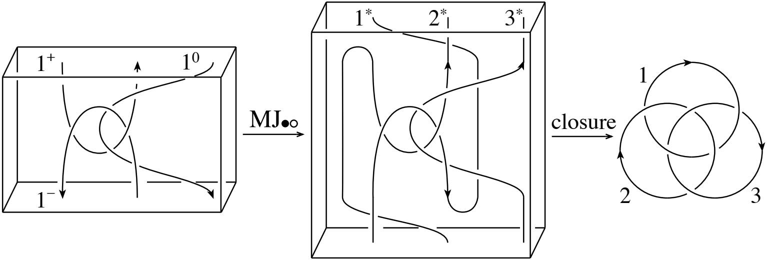
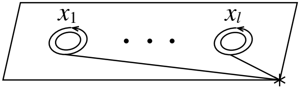


Figure : A bottom-top tangle, the corresponding string link and its closure. Here, $MJ_{\bullet\circ}$ is an extension of the "Milnor-Johnson correspondence" defined in [CHM08].

$$\mathcal{S}_\ell := \{\text{string link } (B, \sigma) \text{ on } \ell \text{ strands} \mid H_*(B) = H_*([-1,1]^3)\}.$$

Milnor invariants

$(B,\sigma) \rightsquigarrow S := \overline{B \setminus N(\sigma)}$ and $s\colon \partial(D_\ell \times [-1,1]) \xrightarrow{\cong} \partial S \hookrightarrow S$.

$D_\ell :=$ 

.

$\varpi := \pi_1(D_\ell, *)$,
$\varpi_1 = \varpi$, $\varpi_k = [\varpi_{k-1}, \varpi]$.

The monoid anti-homomorphism $A_k\colon \mathcal{S}_\ell \to \mathrm{Aut}(\varpi/\varpi_{k+1})$ defined by $A_k(B,\sigma) := s_{+,*}^{-1} \circ s_{-,*}$ is called the kth *Artin representation* ($k \geq 1$).

Definition (Milnor invariant)

The kth *Milnor invariant* is the monoid homomorphism $\mu_k\colon \mathcal{S}_\ell[k](:= \mathrm{Ker}\, A_k) \to \varpi/\varpi_2 \otimes_{\mathbb{Z}} \varpi_k/\varpi_{k+1}$ defined by

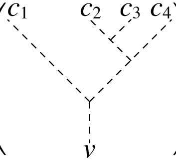
$$\mu_k(B,\sigma) := \sum_{i=1}^{\ell} x_i \otimes s_{+,*}^{-1}(\lambda_i),$$

where λ_i is the longitude of σ.

$\mu_k(B,\sigma)$ $(k \geq 2)$ is regarded as a linear combination $\mu_k^{\mathcal{A}}(B,\sigma)$ of connected tree Jacobi diagrams via the following commutative diagram.

$$\begin{array}{ccccccccc} & & \mathcal{S}_\ell[k] & \xrightarrow{\mu_k} & (\varpi/\varpi_2 \otimes \varpi_k/\varpi_{k+1}) \otimes \mathbb{Q} & & & & \\ & & \Big\downarrow{\scriptstyle \mu_k^{\mathcal{A}}} & & \Big\downarrow{\scriptstyle \cong} & & & & \\ 0 & \longrightarrow & \mathcal{A}_{k-1}^{t,c}(\lfloor \ell \rceil^*) & \xrightarrow{\eta_{k-1}} & \mathbb{Q}\lfloor \ell \rceil^* \otimes_{\mathbb{Q}} \mathrm{Lie}_k(\lfloor \ell \rceil^*) & \xrightarrow{[-,-]} & \mathrm{Lie}_{k+1}(\lfloor \ell \rceil^*) & \longrightarrow & 0 \end{array}$$

$\eta_{k-1}(D) := \sum_v (\text{color of } v) \otimes \mathrm{comm}(D_v)$, where v runs over all external (i.e., trivalent) vertices in D and

comm (

) $:= [c_1, [[c_2, c_3], c_4]]$.

Previous studies and our extension

1. Habegger and Masbaum [HM00] proved that the first non-vanishing Milnor invariant of $([-1,1]^3, \sigma)$ is determined by the first non-trivial term of $(\chi_{\pi_0(\sigma)}^{-1} Z^{\mathrm{K}}(\sigma))^{Y,t}$, and vice versa.
2. Moffatt [Mof06] showed that the same holds for $(\chi_{\pi_0(\sigma)}^{-1} Z^{\text{K-LMO}}(B,\sigma))^{Y,t}$.
3. The same is true for $\widetilde{Z}^{Y,t}(\mathrm{MJ}^{-1}(B,\sigma))$ [CHM08].

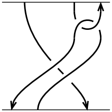
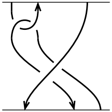
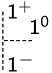
Let $R_w \colon \mathcal{A}(X, \lfloor g\rceil^+ \cup \lfloor g\rceil^- \cup \lfloor n\rceil^0) \xrightarrow{\cong} \mathcal{A}(X, \lfloor \ell\rceil^*)$ be a "color-replacement" map for $w \in \mathrm{Mag}(\bullet, \circ)$.

Theorem (N. '15)

If $(B,\sigma) \in \mathcal{S}_\ell[k]$, *then* $\widetilde{Z}^{Y,t}_{<k}(\mathrm{MJ}_w^{-1}(B,\sigma)) = \varnothing + R_w^{-1}(\mu_k^{\mathcal{A}}(B,\sigma))$.
Conversely, if $\widetilde{Z}^{Y,t}_{<k}(\mathrm{MJ}_w^{-1}(B,\sigma))$ *is written as* $\varnothing + x$ (i-deg $x = k-1$), *then* $\mu_k^{\mathcal{A}}(B,\sigma) = R_w(x)$.

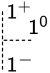
Example ($\ell = 3,\ k = 2,\ \sigma =$ (example in p. 23))

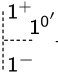
Check $\mathrm{MJ}_{\bullet\circ}^{-1}(\sigma) = \psi_{\circ,\bullet} \circ \psi_{\bullet,\circ}$. Using the functoriality of $\widetilde{Z}$ and

$\widetilde{Z}^Y($  $) = \widetilde{Z}^Y($  $) = \exp_{\sqcup}\Big(\frac{1}{2}$ 

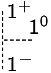
$+ (\text{i-deg} > 2)\Big)$,

we have

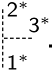
$\widetilde{Z}^Y(\mathrm{MJ}_{\bullet\circ}^{-1}(\sigma)) = \chi_{1^0}^{-1}\chi_{1^0,1^{0'}}\Big(\varnothing + \frac{1}{2}$ 

$+ \frac{1}{2}$ 

$+ (\text{i-deg} > 1)\Big)$

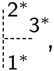
$= \varnothing +$ 

$+ (\text{i-deg} > 1)$.

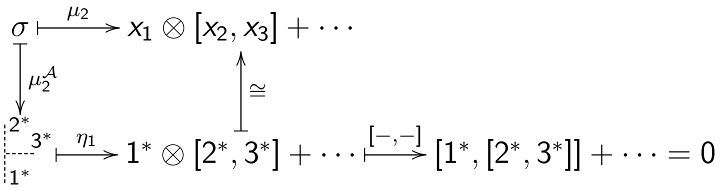
By the previous theorem, $\mu_2^{\mathcal{A}}(\sigma) =$ 

.

Continuation of the previous example

Using the previous result $\mu_2^{\mathcal{A}}(\sigma) =$ 

, we have



where "$\cdots$" denotes the cyclic permutations. It follows that

$$\mu_2(\sigma) = x_1 \otimes [x_2, x_3] + x_2 \otimes [x_3, x_1] + x_3 \otimes [x_1, x_2]$$
$$\in (\varpi/\varpi_2 \otimes \varpi_2/\varpi_3) \otimes \mathbb{Q}.$$

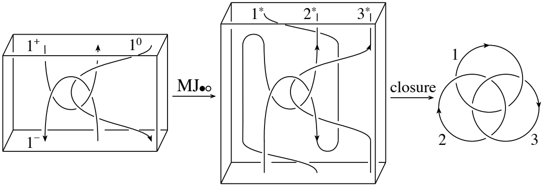
The Milnor $\overline{\mu}$-invariants (of length 3) of the closure of σ is computed as follows:

1. The Magnus expansion of $\mu_2(\sigma) \in \varpi_2/\varpi_3$ is

$$x_1 \otimes (X_2X_3 - X_3X_2 + (\deg > 2)) + (\text{cyclic permutations}).$$

2. Reading the coefficient of $x_1 \otimes X_2X_3$ etc.,

$$\overline{\mu}_{\widehat{\sigma}}(j_1, j_2; i) = \begin{cases} \operatorname{sgn}(j_1\ j_2\ i) & \text{if } \{j_1, j_2, i\} = \{1, 2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$



$$\widetilde{Z}_{<2}(\mathrm{MJ}_{\bullet\circ}^{-1}(\sigma)) \rightsquigarrow \mu_2(\sigma) \rightsquigarrow \overline{\mu}_{\widehat{\sigma}}(j_1, j_2; i)$$

Future research

I would like to investigate the functor $\widetilde{Z}$ and find some applications for it.

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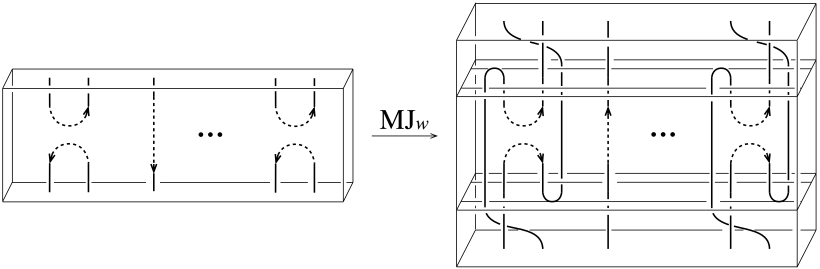

Figure : An extension of the Milnor-Johnson correspondence