Cohomology of Automorphism Groups of Free Groups

Shigeyuki MORITA

based on jw/w Takuya SAKASAI and Masaaki SUZUKI

October 29, 2015

Contents

1. Automorphism groups of free groups vs MCG of surfaces
2. Symplectic derivation Lie algebra and graph homology
3. Conjectural meaning of the Morita classes
4. Prospects

Automorphism groups of free groups vs MCG of surfaces (1)

(Outer) automorphism groups of free groups F_n $(n \geq 2)$

$$1 \to F_n \to \mathrm{Aut}\, F_n \to \mathrm{Out}\, F_n \to 1$$

Mapping class group of surfaces

$$\mathcal{M}_g = \pi_0 \, \mathrm{Diff}^+ \Sigma_g$$

Σ_g : closed oriented surface of genus g (≥ 2)

$\mathcal{T}_g$: Teichmüller space, $\mathcal{M}_g$ acts properly discontinuously

$\mathbf{M}_g = \mathcal{T}_g / \mathcal{M}_g$: **Riemann moduli space**

X_n : Culler-Vogtmann's Outer Space

$\mathbf{G}_n = X_n / \mathrm{Out}\, F_n$: **moduli space of graphs**

Automorphism groups of free groups vs MCG of surfaces (2)

group extensions

$$1 \to \mathrm{IA}_n \xrightarrow{i} \mathrm{Aut}\, F_n \xrightarrow{p} \mathrm{GL}(n, \mathbb{Z}) \to 1$$

IA group

$$1 \to \mathcal{I}_g \to \mathcal{M}_g \underset{\text{action on } H}{\xrightarrow{\rho_1}} \mathrm{Sp}(2g, \mathbb{Z}) \to 1$$

Torelli group **Siegel modular group**

$$H = H_1(\Sigma_g; \mathbb{Z})$$

Automorphism groups of free groups vs MCG of surfaces (3)

stable cohomology

Theorem (Galatius, triviality of the stable cohomology)

$$\lim_{n\to\infty} H^*(\mathrm{Aut}\, F_n;\mathbb{Q}) = \lim_{n\to\infty} H^*(\mathrm{Out}\, F_n;\mathbb{Q}) = \mathbb{Q}$$

stabilize: Hatcher, Hatcher-Vogtmann

Theorem (Madsen-Weiss)

$$\lim_{g\to\infty} H^*(\mathcal{M}_g;\mathbb{Q}) = \mathbb{Q}[\textit{MMM-tautological classes}]$$

Harer stable cohomology

Automorphism groups of free groups vs MCG of surfaces (4)

Theorem (Borel)

$$\lim_{n\to\infty} H^*(\mathrm{GL}(n,\mathbb{Z});\mathbb{R}) = E_{\mathbb{R}}\langle \beta_3, \beta_5, \beta_7, \ldots \rangle$$

$\beta_{2k+1} \in H^{4k+1}(\mathrm{GL}(n,\mathbb{Z});\mathbb{R})$: Borel regulator class

β_{2k+1} vanishes in $H^{4k+1}(\mathrm{Out}\, F_n;\mathbb{R})$ (proved first by Igusa)

Theorem (Borel)

$$\lim_{g\to\infty} H^*(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}) = \mathbb{Q}[c_1, c_3, c_5, \ldots]$$

$c_{2k-1} \in H^{4k-2}(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q})$: Chern class

$c_{2k-1} \in H^{2k}(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}) \Rightarrow$ poly. on $e_{\mathrm{odd}} \in H^{2k}(\mathcal{M}_g;\mathbb{Q})$

Symplectic derivation Lie algebra and graph homology (1)

$$H_{\mathbb{Q}} = H \otimes \mathbb{Q} = H_1(\Sigma_g;\mathbb{Q}), \quad \mathcal{M}_{g,1} = \pi_0 \operatorname{Diff}(\Sigma_g, D^2)$$

Theorem (Dehn-Nielsen-Zieschang)

- $\mathcal{M}_g \cong \mathrm{Out}^+\pi_1\Sigma_g$ *(outer automorphism group)*
- $\mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\,\pi_1\Sigma_{g,1}; \varphi(\zeta) = \zeta\}$ ζ : *boundary curve*

$$\Sigma_{g,1} = \Sigma_g \setminus \operatorname{Int} D^2$$

"differentiate" $\Rightarrow$

Definition (symplectic derivation Lie algebra)

$\mathfrak{h}_{g,1} = \{$symplectic derivation of the free Lie algebra $\mathcal{L}(H_{\mathbb{Q}})\}$

greded Lie algebra, $\mathfrak{h}_{g,1} \supset \mathfrak{h}_{g,1}^+$: ideal of positive derivations

Symplectic derivation Lie algebra and graph homology (2)

Mal'cev nilpotent completion of $\pi_1\Sigma_{g,1}$:

$$\cdots \to N_{d+1} \to N_d \to \cdots \to N_1 = H_{\mathbb{Q}} \to 0 \quad (H_{\mathbb{Q}} = H_1(\Sigma_g;\mathbb{Q}))$$

$\Rightarrow$ obtain a series of representations of $\mathcal{M}_{g,1}$:

$$\rho_\infty = \{\rho_d\}_d : \mathcal{M}_{g,1} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \quad (\rho_d : \mathcal{M}_{g,1} \to \mathrm{Aut}_0\, N_d)$$

associated embedding of graded Lie algebras:

$$\tau : \bigoplus_{d=1}^{\infty} \mathcal{M}_{g,1}(d)/\mathcal{M}_{g,1}(d+1) \overset{\text{small}}{\subset} \mathfrak{h}_{g,1}^{+} \overset{\text{ideal}}{\subset} \mathfrak{h}_{g,1}$$

$\mathcal{M}_{g,1}(d) := \mathrm{Ker}\,\rho_d$ Johnson filtration

Determination of $\mathrm{Im}\,\tau$: still open and very difficult

Many works about $\mathrm{Coker}\,\tau \overset{\text{Hain}}{=} \mathfrak{h}^+_{g,1}/\langle\wedge^3 H_{\mathbb{Q}}\rangle$:

- Morita traces (generators , i.e. survives in $H_1(\mathfrak{h}^+_{g,1})$)
- Galois images: Nakamura, Matsumoto (decomposable?)
- Enomoto-Satoh traces (decomposable), Kawazumi-Kuno
- New generators+others: Conant-Kassabov-Vogtmann
- Conant (still more new generators)
- Conant-Kassabov ...

Lie version of Kontsevich graph homology

Theorem (Kontsevich, Lie version)

There exists an isomorphism

$$PH_c^k(\widehat{\mathfrak{h}}_{\infty,1})_{2n} \cong H_{2n-k}(\mathrm{Out}\, F_{n+1}; \mathbb{Q})$$

$\widehat{\mathfrak{h}}_{\infty,1}$: completion of $\mathfrak{h}_{\infty,1} = \lim_{g\to\infty} \mathfrak{h}_{g,1}$

$$\bigoplus_{n\geq 2} H_*(\mathrm{Out}\, F_n; \mathbb{Q}) \Leftrightarrow PH_c^*(\widehat{\mathfrak{h}}_{\infty,1})$$

equivalent!

Symplectic derivation Lie algebra and graph homology (5)

$$\bigoplus_{n\geq 2} H_{2n-3}(\mathrm{Out}\, F_n;\mathbb{Q}) \Leftrightarrow PH^1_c(\widehat{\mathfrak{h}}_{\infty,1}) \overset{\text{dual}}{\Leftrightarrow} H_1(\mathfrak{h}^+_{\infty,1})_{\mathrm{Sp}}$$

Culler-Vogtmann: $\mathrm{vcd}(\mathrm{Out}\, F_n) = 2n-3$

Problem

What are the generators: $H_1(\mathfrak{h}^+_{\infty,1})$ for the Lie algebra $\mathfrak{h}^+_{\infty,1}$?

$$\bigoplus_{n\geq 2} H_{2n-4}(\mathrm{Out}\, F_n;\mathbb{Q}) \Leftrightarrow PH^2_c(\widehat{\mathfrak{h}}_{\infty,1})$$

Problem

What is the second cohomology of the Lie algebra $\mathfrak{h}_{\infty,1}$?

Generators for $\mathfrak{h}^+_{g,1}$ $(= H_1(\mathfrak{h}^+_{g,1}))$:

$\wedge^3 H_{\mathbb{Q}} = \mathfrak{h}_{g,1}(1)$ Johnson

traces: $\displaystyle\bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}$ Morita

Theorem (Conant-Kassabov-Vogtmann)

$H_1(\mathfrak{h}^+_{g,1}) \cong \wedge^3 H_{\mathbb{Q}}$ (*Johnson, 0-loop*)
$\oplus \left(\oplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}\right)$ (*M., trace maps: 1-loop*)
$\oplus \left(\oplus_{k=1}^{\infty} [2k+1, 1]_{\mathrm{Sp}} \oplus\right.$ *other part*$)$ (2-*loops*)
$\oplus$ *non-trivial ?* ($3, 4, \ldots$-*loops*) *?* : *deep question*

very recently, Conant: 3-loops non-trivial

Construction of elements of $H^2_c(\widehat{\mathfrak{h}}_{\infty,1})$

trace maps : $\mathfrak{h}^+_{g,1} \to \bigoplus_{k=1}^{\infty} S^{2k+1}H_{\mathbb{Q}}, \ H^2(S^{2k+1}H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \Rightarrow$

$$\mathbf{t}_{2k+1} \in H^2_c(\widehat{\mathfrak{h}}_{\infty,1})_{4k+2} \overset{K.}{\cong} H_{4k}(\mathrm{Out}\, F_{2k+2};\mathbb{Q})$$

$\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2};\mathbb{Q}) \ (k=1,2,\ldots)$ Morita classes

Theorem (non-triviality of μ_k)

$\mu_1 \neq 0 \in H_4(\mathrm{Out}\, F_4;\mathbb{Q})$ (*M. 1999*)

$\mu_2 \neq 0 \in H_8(\mathrm{Out}\, F_6;\mathbb{Q})$ (*Conant-Vogtmann 2004*)

$\mu_3 \neq 0 \in H_{12}(\mathrm{Out}\, F_8;\mathbb{Q})$ (*Gray 2011*)

Symplectic derivation Lie algebra and graph homology (8)

$H^*(\mathrm{Out}\, F_n;\mathbb{Q}) \cong H^*(\mathbf{G}_n;\mathbb{Q})$ characteristic classes of

moduli space of graphs (Culler-Vogtmann)

computed for $n \leq 6$, only two non-trivial parts

$H_4(\mathrm{Out}\, F_4;\mathbb{Q}) \cong \mathbb{Q}$ (Hatcher-Vogtmann)

$H_8(\mathrm{Out}\, F_6;\mathbb{Q}) \cong \mathbb{Q}$ (Ohashi)

Conjecture (very difficult and important)

$\mu_k \neq 0$ *for all* k $\left(\Rightarrow H^2(\mathfrak{h}_{\infty,1}) \supset \mathbb{Q}\langle e_1, \mathfrak{t}_3, \mathfrak{t}_5, \cdots\rangle\right)$

New approach (assembling homology classes) , in particular;

Theorem (Conant-Hatcher-Kassabov-Vogtmann)

The class μ_k is supported on certain subgroup $\mathbb{Z}^{4k} \subset \mathrm{Out}\, F_{2k+2}$

CKV new generators $\Rightarrow$ more classes in $H^2_c(\widehat{\mathfrak{h}}_{\infty,1})$

Many odd dimensional cohomology classes exist:

Theorem (Sakasai-Suzuki-M.)

The integral Euler characteristics of $\mathrm{Out}\, F_n$ *is given by*

$$e(\mathrm{Out}\, F_n) = 1, 1, 2, 1, 2, 1, 1, -21, -124, -1202 \ (n = 2, 3, \ldots, 11)$$

No explicit one is known

Problem

Construct non-trivial odd dim. homology classes of $\mathrm{Out}\, F_n$

Conjectural meaning of the Morita classes (1)

Conjectural geometric meaning of the classes

$$\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q})$$

secondary classes associated with the difference between

two reasons for the vanishing of Borel regulator classes

$$\beta_{2k+1} \in H^{4k+1}(\mathrm{GL}(N, \mathbb{Z}); \mathbb{R}) \ (k = 1, 2, 3, \dots)$$

(1) $\beta_{2k+1} = 0 \in H^{4k+1}(\mathrm{Out}\, F_N; \mathbb{R})$ (Igusa, Galatius)

(2) $\beta_{2k+1} = 0 \in H^{4k+1}(\mathrm{GL}(N_k^*, \mathbb{Z}); \mathbb{R})$ critical $N_k^* \overset{?}{=} 2k+2$

yes: $k = 1$ (Lee-Szczarba), $k = 2$ (E.Vincent-Gangl-Soulé)

$\beta_{2k+1} \neq 0 \in H^{4k+1}(\mathrm{GL}(2k+3, \mathbb{Z}); \mathbb{R})$ (announced by Lee)

Theorem (Bismut-Lott, Lee, Franke)

$$\beta_{2k+1} = 0 \in H^{4k+1}(\mathrm{GL}(2k+1, \mathbb{Z}); \mathbb{R})$$

Conjectural meaning of the Morita classes (2)

Strategy of a proof of the conjecture:

secondary classes associated with the difference between two reasons for β_{2k+1} to vanish

$b_{2k+1} \in Z^{4k+1}(\mathrm{GL}(N,\mathbb{Z});\mathbb{R})$ cocycle, e.g. Hamida's cocycle

(1) $p^*(b_{2k+1}) = \delta z_{4k}\ (z_{4k} \in C^{4k}(\mathrm{Aut}\, F_N;\mathbb{R}))$

(2) $i^*(b_{2k+1}) = \delta z'_{4k}\ (z'_{4k} \in C^{4k}(\mathrm{GL}(2k+2,\mathbb{Z});\mathbb{R}))$

Conjecture

$$\langle i^*(z_{4k}) - p^*(z'_{4k}), \mu_k\rangle \neq 0 \quad (\textit{if yes} \ \Rightarrow\ \mu_k \neq 0)$$

"dual version" : $\langle$Hamida cocycle, certain $4k+1$ cycle$\rangle \neq 0$

planning computer computation

Conjectural meaning of the Morita classes (3)

Comparison with the case of the Casson invariant

$\lambda(M) \in \mathbb{Z} \quad (M$: homology 3-sphere)

interpreted as a homomorphism

$$d_1 : \mathcal{K}_g \to \mathbb{Z} \quad (\mathcal{K}_g\text{: Johnson kernel})$$

$$d_1 \in H^1(\mathcal{K}_g;\mathbb{Z})^{\mathcal{M}_g} \cong \mathbb{Z} : \text{ generator}$$

secondary classes associated with the difference between two cocycles for the first MMM class $\in H^2(\mathcal{M}_g;\mathbb{Z}) \cong \mathbb{Z}$

(1) Meyer's cocycle for $c_1 \in H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q})$

(2) "intersection cocycle" defined by using

$$\tilde{k} \in H^1(\mathcal{M}_g;\wedge^3 H/H) \cong \mathbb{Q} \ (g \geq 3)$$

Conjectural meaning of the Morita classes (4)

Related secondary classes associated to the vanishing of the Borel classes on $\mathrm{Out}\, F_N, \mathrm{Aut}\, F_N$

$$p^*(b_{2k+1}) = \delta z_{4k} \quad (z_{4k} \in C^{4k}(\mathrm{Aut}\, F_N; \mathbb{R}), C^{4k}(\mathrm{Out}\, F_N; \mathbb{R}))$$

$$\delta i^*(z_{4k}) = 0 \ \Rightarrow [i^*(z_{4k})] \in H^{4k}(\mathrm{IA}_N; \mathbb{R}), H^{4k}(\mathrm{IOut}_N; \mathbb{R})$$

Galatius' theorem $\Rightarrow$ in the stable range, this class well-defined $\mathrm{GL}(N, \mathbb{Z})$-invariant, because for any $\varphi \in \mathrm{Aut}\, F_N$, can show

$$\iota_\varphi^*(i^*(z_{4k})) \overset{\text{cohomologous}}{\sim} i^*(z_{4k}) \quad (\iota_\varphi\text{: conjugation by } \varphi)$$

Definition (stable secondary class)

$$T\beta_{2k+1} = [i^*(z_{4k})] \in H^{4k}(\mathrm{IOut}_N \text{ or } \mathrm{IA}_N, ; \mathbb{R})^{\mathrm{GL}(N,\mathbb{Z})}$$

Theorem

$T\beta_{2k+1} =$ *Igusa's higher* FR *torsion class* $\tau_{2k} \in H^{4k}(\mathrm{IOut}_N; \mathbb{R})$

Conjectural meaning of the Morita classes (5)

If we consider the spectral sequence for

$$\mathrm{IOut}_N \to \mathrm{Out}\, F_N \to \mathrm{GL}(N,\mathbb{Z})$$

$\beta_{2k+1} \in H^{4k+1}(\mathrm{GL}(N,\mathbb{Z});\mathbb{R}) = E_2^{4k+1,0}$ must be killed by some of $H^{4k-i}(\mathrm{GL}(N,\mathbb{Z}); H^i(\mathrm{IOut}_N;\mathbb{R}))$ $(i = 1, \ldots, 4k)$ and $T\beta_{2k+1} \in H^0(\mathrm{GL}(N,\mathbb{Z}); H^{4k}(\mathrm{IOut}_N;\mathbb{R})) = E_2^{0,4k}$

Theorem (Hain-Igusa-Penner)

The higher FR *torsion of the Torelli group is a non-zero multiple of the even* MMM *class*

Conjecture (Church-Farb)

$$\lim_{N\to\infty} \widetilde{H}^*(\mathrm{IA}_N;\mathbb{Q})^{\mathrm{GL}(N,\mathbb{Z})} = 0$$

Conjectural meaning of the Morita classes (6)

If yes $\Rightarrow$ all the even MMM classes vanish on the Torelli group

We would like to propose another

(so to speak "opposite" and perhaps too optimistic?) possibility:

Conjecture

$$\lim_{N \to \infty} H^*(\mathrm{IOut}_N; \mathbb{R})^{\mathrm{GL}(N,\mathbb{Z})} \cong \mathbb{R}[\tau_2, \tau_4, \cdots]$$

Our conjecture on geometric meaning of μ_k can be interpreted

as the farmost unstable version of the above conjecture

these two conjectures are closely related but independent

Prospects (1)

Conjecture

$$H_1(\mathfrak{h}_{\infty,1}) \ \left(\cong H_1(\mathfrak{h}^+_{\infty,1})_{\mathrm{Sp}}\right) = 0$$

Yes $\overset{\text{Kontsevich}}{\Leftrightarrow}$ $H^{2n-3}(\mathrm{Out}\, F_n;\mathbb{Q}) = 0$ for any $n \geq 2$

Theorem (Sakasai-Suzuki-M. associative case)

$$H_1(\mathfrak{a}^+_\infty) \cong \lim_{g\to\infty} \wedge^3 H_{\mathbb{Q}} \oplus S^3 H_{\mathbb{Q}} \oplus \left(\wedge^2 H_{\mathbb{Q}}/\langle\omega_0\rangle\right) \Rightarrow \ H_1(\mathfrak{a}_\infty) = 0$$

Corollary (vanishing of the top cohomology)

$$H^{4g-5}(\mathcal{M}_g;\mathbb{Q}) = 0 \quad (g \geq 2)$$

Harer (unpublished), another proof: Church-Farb-Putman

Prospects (2)

Possible (but conjecturally no) contribution on $H_1(\mathfrak{h}^+_{\infty,1})_{\mathrm{Sp}}$

(I) Arithmetic mapping class group

(II) Group of H-cobordism classes of homology cylinders

(I) Arithmetic mapping class group

$$1 \to \widehat{\mathcal{M}_g^1} \to \pi_1^{\mathrm{alg}}(\mathbf{M}_g^1/\mathbb{Q}) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1$$

$\widehat{\mathcal{M}_g^1}$: profinite completion of $\mathcal{M}_g^1 = \pi_0 \operatorname{Diff}^+(\Sigma_g, *)$

Grothendieck, Deligne, Ihara, Drinfel'd,...

1980's, Oda predicted: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (Soulé p-adic regulators) should "appear" in $(\mathrm{Coker}\,\tau)^{\mathrm{Sp}} \otimes \mathbb{Z}_p$ (p: prime)

Nakamura, Matsumoto: proof and related many works

precise description of the Galois images: unknown

important both in topology and number theory

Conjecture (around 1984)

The Galois images are decomposable and can be described in terms of the M. traces

If yes $\Rightarrow$ Galois images do *not* survive in $H_1(\mathfrak{h}^+_{\infty,1})_{\mathrm{Sp}}$

Prospects (4)

Fundamental Lie algebra $\mathfrak{f} := \text{Free Lie}\langle \sigma_3, \sigma_5, \cdots \rangle$

$\mathfrak{f} \overset{\subseteq}{\to} \mathfrak{h}_{g,1}^{\mathrm{Sp}}$ Nakamura, Matsumoto, ... , Brown

$\mathfrak{f} \overset{\subseteq}{\to} \mathfrak{h}_{1,1}^{\mathrm{Sp}}$ Hain-Matsumoto, Pollack

Problem

Describe the image of σ_k explicitly in each case

Theorem (Sakasai-Suzuki-M.)

We have determined the structure of $\mathfrak{h}_{g,1}(6)$ $(\ni \sigma_3)$ completely

Recent progress by Hain (with Brown, Matsumoto)

Prospects (5)

(II) Group of H-cobordism classes of homology cylinders

Garoufalidis-Levine (based on Goussarov and Habiro):

$$\mathcal{H}_{g,1} = \{(\textbf{homology } \Sigma_{g,1} \times I, \varphi); \varphi : \Sigma_{g,1} \cong \Sigma_{g,1} \times \{1\}\} \Big/ \text{homology cobordism}$$

two versions: smooth $\mathcal{H}_{g,1}^{\text{smooth}}$ and topological $\mathcal{H}_{g,1}^{\text{top}}$

enlargement of $\mathcal{M}_{g,1}$:

$$\mathcal{M}_{g,1} \ni \varphi \mapsto (\Sigma_{g,1} \times I, \varphi) \in \mathcal{H}_{g,1}^{\text{smooth}}, \mathcal{H}_{g,1}^{\text{top}}$$

Prospects (6)

Theorem (Garoufalidis-Levine, Habegger)

There exists a homomorphism

$$\tilde{\rho}_\infty : \mathcal{H}_{g,1} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d$$

which extends ρ_∞*, each finite factor* $\tilde{\rho}_d : \mathcal{H}_{g,1} \to \mathrm{Aut}_0\, N_d$ *is surjective over* $\mathbb{Z}$ *for any* $d \geq 1$

$$\begin{array}{ccc} \mathcal{M}_{g,1}(d) & \xrightarrow[\text{image small}]{\tau_d} & \mathfrak{h}_{g,1}(d) \\ \cap\downarrow & & \| \\ \mathcal{H}_{g,1}(d) & \xrightarrow[\text{surjective}]{\tilde{\tau}_d} & \mathfrak{h}_{g,1}(d) \end{array}$$

$\mathfrak{h}_{g,1}$: too big as Lie algebra for $\mathcal{M}_{g,1}$, how about for $\mathcal{H}_{g,1}$? No!

$$
\begin{array}{ccc}
\mathcal{M}_{g,1} & \xrightarrow[\text{injective}]{\rho_\infty} & \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d \\
\cap \downarrow & & \| \\
\mathcal{H}^{\mathrm{smooth}}_{g,1} & \xrightarrow[\text{Not injective}]{\tilde{\rho}_\infty} & \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d
\end{array}
$$

Theorem (Sakasai, Dehn-Nielsen type theorem for $\mathcal{H}_{g,1}$)

$\mathcal{H}^{\mathrm{top}}_{g,1} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d$ *factors through*

$\mathcal{H}^{\mathrm{alg}}_{g,1} = \mathrm{Aut}_0\, F^{\mathrm{acy}}_{2g}$ *: Sakasai's algebraic version*

by using acyclic closure F^{acy}_{2g} of Levine

works for $\mathcal{H}_{g,1}$: survey by Sakasai and Habiro-Massuyeau

Prospects (8)

Define a quotient group $\overline{\mathcal{H}}_{g,1}$ by the following central extension

$$0 \to \mathbf{\Theta}^3 = \mathcal{H}_{0,1} \to \mathcal{H}_{g,1}^{\text{smooth}} \to \overline{\mathcal{H}}_{g,1} \to 1$$

$\mathbf{\Theta}^3 :=$ Homology cobordism group of homology 3-spheres

infinite rank by Furuta, Fintushel-Stern, also $\tilde{\rho}_\infty(\mathbf{\Theta}^3) = \{1\}$

Prospects (8)

Define a quotient group $\overline{\mathcal{H}}_{g,1}$ by the following central extension

$$0 \to \mathbf{\Theta}^3 = \mathcal{H}_{0,1} \to \mathcal{H}_{g,1}^{\text{smooth}} \to \overline{\mathcal{H}}_{g,1} \to 1$$

$\mathbf{\Theta}^3 :=$ Homology cobordism group of homology 3-spheres

infinite rank by Furuta, Fintushel-Stern, also $\tilde{\rho}_\infty(\mathbf{\Theta}^3) = \{1\}$

Problem

Study the Euler class

$$\chi(\mathcal{H}_{g,1}^{\text{smooth}}) \in H^2(\overline{\mathcal{H}}_{g,1}; \mathbf{\Theta}^3)$$

$$\mathbf{\Theta}^3 \to \mathcal{H}_{g,1}^{\text{smooth}} \to \overline{\mathcal{H}}_{g,1} \xrightarrow{\text{Freedman}} \mathcal{H}_{g,1}^{\text{top}} \to \mathrm{Aut}_0\, F_{2g}^{\text{acy}} \to \varprojlim_d \mathrm{Aut}_0\, N_d$$

Sakasai

Prospects (9)

One of the foundational results of Freedman:

Theorem (Freedman)

Any homology 3-sphere bounds a contractible topological 4-manifold so that $\Theta^3(top) = 0$

It follows that $\mathcal{H}_{g,1}^{\mathrm{smooth}} \to \mathcal{H}_{g,1}^{\mathrm{top}}$ factors through $\overline{\mathcal{H}}_{g,1}$

Problem (about "Picard groups")

Study the following homomorphisms

$$H^2(\mathcal{H}_{g,1}^{\mathrm{top}}) \to H^2(\overline{\mathcal{H}}_{g,1}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}}) \to H^2(\mathcal{M}_{g,1}) \overset{\mathrm{Harer}}{\cong} \mathbb{Z}$$

∞-rank? ∞-rank? $\cong$?

Problem

Determine whether $H_1(\mathcal{H}_{g,1}^{\text{smooth}};\mathbb{Q}) = 0$ *or not*

Theorem (Cha-Friedl-Kim)

$H_1(\mathcal{H}_{g,1}^{\text{smooth}})$ *contains* $(\mathbb{Z}/2)^\infty$ *as a direct summand*

$H^2(\mathcal{H}_{g,1}^{\text{smooth}})$ versus $H^2(\mathcal{H}_{g,1}^{\text{top}})$, mystery in dimension 4

Theorem (M., stable homomorphism w. Zariski dense image)

$$\tilde{\rho} : \mathcal{H}_{g,1}^{\text{top}} \longrightarrow \left(\wedge^3 H_{\mathbb{Q}} \times \prod_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

Massuyeau-Sakasai

$\tilde{\rho}^*$ on H^* yields many stable cohomology classes of $\mathcal{H}_{g,1}^{\mathrm{top}}$

Corollary

The MMM-classes are defined already in $H^*(\mathcal{H}_{g,1}^{\mathrm{top}}, \mathbb{Q})$

Definition (characteristic classes for homology cylinders)

$$\tilde{\mathbf{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1}; \mathbb{Q}), H^2(\mathcal{H}_{g,1}^{\mathrm{top}}; \mathbb{Q}) \quad (k = 1, 2, \dots)$$

most important classes coming from $H^2(S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q}$

candidates for $\chi(\mathcal{H}_{g,1}^{\mathrm{smooth}}) \in H^2(\overline{\mathcal{H}}_{g,1}; \mathbf{\Theta}^3)$, group version of

$$\mathbf{t}_{2k+1} \in H^2(\mathfrak{h}_{g,1}; \mathbb{Q})_{4k+2} \cong H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q}) \ni \mu_k$$

geometrical meaning of the classes $\tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q})$:
Intersection numbers of higher and higher Massey products
(using works of Kitano, Garoufalidis-Levine)

Conjecture

In the central extension

$$0 \to \mathbf{\Theta}^3 \to \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \overline{\mathcal{H}}_{g,1} \to 1$$

$\mathbf{\Theta}^3$ *"transgresses" to the classes* $\tilde{\mathbf{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \Rightarrow$

$$\tilde{\mathbf{t}}_{2k+1} \neq 0 \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}), H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q}) \text{ and}$$
$$\tilde{\mathbf{t}}_{2k+1} = 0 \in H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})$$

If yes $\Rightarrow$ obtain homomorphisms $\nu_k : \mathbf{\Theta}^3 \to \mathbb{Z}$