On the Θ-invariants of 3-manifolds

('19/6/7, Kanazawa)

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科研費
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background

Chern-Simons q.f.t (Witten) '89

$\downarrow$ perturbative expansion

$\boxed{\text{Chern-Simons pert. theory} \begin{pmatrix} \text{Kontsevich} \\ \text{Axelrod-Singer} \end{pmatrix}}$ '91

→ invariants of (a framed 3-mfd, a local system on the mfd.)

- trivial local system ⋯ universal finite type inv.s
  (Kuperberg-Thurston, Taubes, Lescop, ⋯)
- Non-trivial local system ⋯ $\boxed{\text{Bott-Cattaneo's inv.}}$
  (degree 1 term : "Θ-invariant.")

background

Chern-Simons q.f.t (Witten)

$\downarrow$ perturbative expansion

$\boxed{\text{Chern-Simons pert. theory } \left(\begin{array}{l}\text{Kontsevich}\\ \text{Axelrod-Singer}\end{array}\right)}$

→ invariants of (a framed 3-mfd, a local system on the mfd.)

- trivial local system ... universal finite type inv.s
  ($\mathbb{R}$) (Kuperberg-Thurston, Taubes, Lescop, ...)
- Non-trivial local system ... $\boxed{\text{Bott-Cattaneo's inv.}}$

Today → "~~gap~~" remove $\in$ (degree 1 term : "Θ-invariant.")

j.w.w/ A. Cattaneo

Plan of this talk

1. $M = \mathbb{R}^3$, trivial local system $\mathbb{R}$

2. M : a punctured $\mathbb{Q}HS^3$, triv. loc. system $\mathbb{R}$
   $(\to M : \mathbb{Q}H\mathbb{R}^3)$

3. M : a closed 3-mfd, E : (non-trivial) loc. system
   (Bott-Cattaneo's original construction)

4. (Main result) Correction of 3. (j. w. w/ A. S. Cattaneo)

(5. "higher" inv.)

degree 1 term of C.S. pert theory (Θ-inv.) (outline)

input $\begin{cases} M : \text{a framed 3-mfd} \\ E : \text{a local system} \end{cases}$ i.e. $TM \xrightarrow{\cong} M \times \mathbb{R}^3$

(s.t. $H^i(M;E) = 0,\ i \geq 1$ + some conditions)

$\rightsquigarrow$

$\exists$ propagator $\omega \in \Omega^2(M^2 \setminus \underbrace{\Delta}_{\text{diagonal}} ; E \boxtimes E)$ closed 2-form

$\rightsquigarrow$

configuration space integral "$\int_{M^2 \setminus \Delta} \omega^3$" $\in \mathbb{R}$

$\rightsquigarrow$

Output $I_\Theta(M, E) =$ "$\int_{M^2 \setminus \Delta} \omega^3$" $\in \mathbb{R}$.

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1. $M = \mathbb{R}^3$ ($=$ punctured S^3), $\underline{\mathbb{R}}$ (triv. loc. sys.)

(framing : $TM = \mathbb{R}^3 \times \mathbb{R}^3$ (standard framing))

$\to H^i(M; \underline{\mathbb{R}}) = 0 \quad (i \geq 1)$

- Let $\varphi : M^2 \setminus \Delta \;(= \{(x,y) \mid x \neq y \in M\}) \longrightarrow S^2 \;(\subset \mathbb{R}^3)$

$$(x,y) \longmapsto \frac{y-x}{\|y-x\|} \quad \text{(Gauß map)}$$

- Take $\omega_{S^2} \in \Omega^2(S^2)$; volume form, $\left(\int_{S^2} \omega_{S^2} = 1 \right)$

$\longrightarrow \underbrace{\varphi^* \omega_{S^2}}_{\text{propagator}} \in \Omega^2(M^2 \setminus \Delta)$, closed 2-form

$\omega :=$

def $\quad I_\Theta(M) = \int_{M^2 \setminus \Delta} \omega^3 \in \mathbb{R}$

(Θ-inv. of M $(=\mathbb{R}^3)$)

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Remark : geometric meaning of ω (propagator)

$\omega \in \Omega^2(M^2 \setminus \Delta)$, propagator

$\forall$ $K_1 \sqcup K_2 \subset M \ (= \mathbb{R}^3)$, 2-comp. link.

$\longrightarrow K_1 \times K_2 \subset M^2 \setminus \Delta$, a torus.

fact

$$\int_{K_1 \times K_2} \omega \big|_{K_1 \times K_2} = lk(K_1, K_2)$$

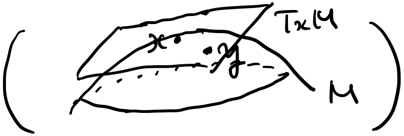
2. M: a framed punctured $\mathbb{Q}HS^3$, $\underline{\mathbb{R}}$ (triv. loc. sys)

$\to H^i(M;\underline{\mathbb{R}}) = 0 \quad (i \geq 1)$

~~• $\varphi : M \times M \setminus \Delta \longrightarrow S^2$, $(x,y) \longmapsto \dfrac{y-x}{\|y-x\|}$??~~

• Let $N(\Delta) \subset M^2$: small neighborhood of Δ in M^2.

• $(x,y) \in N(\Delta) \setminus \Delta \iff x \sim y \rightsquigarrow y \text{ "}\in\text{" } T_xM \underset{\uparrow\,\text{framing}}{\cong} \mathbb{R}^3$



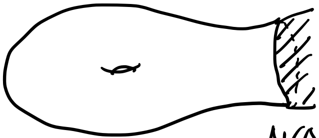
$\rightsquigarrow \dfrac{y-x}{\|y-x\|}$ makes sense !!

$\to \varphi_\Delta : N(\Delta) \setminus \Delta \longrightarrow S^2, \quad (x,y) \longmapsto \dfrac{y-x}{\|y-x\|}$.

$$\begin{array}{ccc} \to \varphi_\Delta^{*}\omega_{S^2} & \in & \Omega^2(N(\Delta)\setminus\Delta),\ \text{closed} \\ \Big\downarrow \text{extend} & & \Big\uparrow \\ \underset{\text{propagator}}{\omega} & \in & \Omega^2(M^2\setminus\Delta),\ \text{closed} \end{array}$$

Propagator

Lem. & def. ① (existence) $\exists\, \omega \in \Omega^2(M^2 \setminus \Delta)$; propagator



s.t.
- $d\omega = 0$
- $\omega|_{N(\Delta)\setminus\Delta} = \varphi_\Delta^* \omega_{S^2}$ (boundary condition)
- (• conditions on near punctured pt)

② (uniqueness) ω is unique as a cohomology class.

i.e. $\forall \omega'$: alt. choice, $[\omega - \omega'] = 0 \in H^2(M^2 \setminus \Delta, N(\Delta)\setminus\Delta)$

proof

$H^2(M;\mathbb{R}) = 0 \Rightarrow$ ① (existence)

$H^1(M;\mathbb{R}) = 0 \Rightarrow$ ② (uniqueness)

//

Definition of the invariant

$$\left(\begin{array}{l} \varphi_\Delta : N(\Delta)\setminus\Delta \to S^2 \\ \varphi_\Delta^* \omega_{S^2} \in \Omega^2(N(\Delta)\setminus\Delta) \\ \quad \big\downarrow \text{extend} \\ \omega \in \Omega^2(M^2\setminus\Delta),\ \text{closed} \\ \text{propagator} \end{array}\right)$$

<u>def</u> $$I_\theta(M) = \int_{M^2\setminus\Delta} \omega^3 \in \mathbb{R}$$

(Θ-invariant of M)

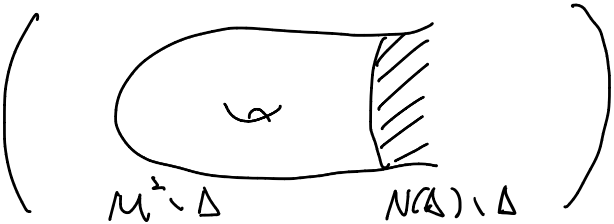
Well-def. of $I_\Theta(M)$

Thm. (Taubes, Kuperberg-Thurston, Lescop, ⋯)
$I_\Theta(M)\left(=\int_{M^2\setminus\Delta}\omega^3\right)$ is inv. of M (i.e. indep of the choice of ω)

(proof) $\omega_0, \omega_1 \in \Omega^2(M^2\setminus\Delta)$, propagators.

- $\longrightarrow$ (by the def) $\omega_0 = \omega_1 = \varphi_\Delta^* \omega_{S^2}$ on $N(\Delta)\setminus\Delta$.
- "uniquness" $\left([\omega_0-\omega_1]=0 \in H^2(M^2\setminus\Delta, N(\Delta)\setminus\Delta)\right)$

  $\Rightarrow \exists\, \eta \in \Omega^1(M^2\setminus\Delta, N(\Delta)\setminus\Delta)$ s.t. $d\eta = \omega_0 - \omega_1$.
- $\displaystyle\int_{M^2\setminus\Delta}\omega_0^3 - \int_{M^2\setminus\Delta}\omega_1^3 = \int_{M^2\setminus \underline{\underline{N(\Delta)}}} \omega_0^3-\omega_1^3 = \int_{M^2\setminus N(\Delta)} (\omega_0-\omega_1)(\omega_0^2+\omega_0\omega_1+\omega_1^2)$



Well-def. of $I_\Theta(M)$

$\boxed{\begin{array}{l}\underline{\text{Thm.}}\ (\text{Taubes, Kuperberg-Thurston, Lescop}, \cdots) \\ I_\Theta(M) \left(= \int_{M^2\setminus\Delta} \omega^3 \right) \text{ is inv. of } M \ (\text{i.e. indep of the choice of } \omega)\end{array}}$

(proof) $\omega_0, \omega_1 \in \Omega^2(M^2 \setminus \Delta)$, propagators.

- $\longrightarrow$ (by the def) $\omega_0 = \omega_1 = \varphi_\Delta^* \omega_{S^2}$ on $N(\Delta)\setminus\Delta$.
- "uniquness" ($[\omega_0 - \omega_1] = 0 \in H^2(M^2\setminus\Delta, N(\Delta)\setminus\Delta)$)

  $\Rightarrow \exists \eta \in \Omega^1(M^2\setminus\Delta, N(\Delta)\setminus\Delta)$ s.t. $d\eta = \omega_0 - \omega_1$.

- $\displaystyle \int_{M^2\setminus\Delta} \omega_0^3 - \int_{M^2\setminus\Delta} \omega_1^3 = \int_{M^2\setminus N(\Delta)} \omega_0^3 - \omega_1^3 = \int_{M^2\setminus N(\Delta)} \underbrace{(\omega_0 - \omega_1)}_{\text{"}d\eta}(\omega_0^2 + \omega_0\omega_1 + \omega_1^2)$

$$= \int_{M^2\setminus N(\Delta)} d\left(\eta(\omega_0^2 + \omega_0\omega_1 + \omega_1^2)\right) \underset{\text{Stokes' thm}}{=} \int_{\partial N(\Delta)} (\eta|_{\partial N(\Delta)})\left(3 \cdot (\varphi_\Delta^* \omega_{S^2}^2 |_{\partial N(\Delta)})\right)$$

Well-def. of $I_\Theta(M)$

Thm. (Taubes, Kuperberg-Thurston, Lescop, …)

$I_\Theta(M)$ $\left(=\int_{M^2\setminus\Delta}\omega^3\right)$ is inv. of M (i.e. indep of the choice of ω)

(proof) $\omega_0, \omega_1 \in \Omega^2(M^2\setminus\Delta)$, propagator

- $\longrightarrow$ (by the def) $\omega_0 = \omega_1 = \varphi_\Delta^*\omega_{S^2}$ on $N(\Delta)\setminus\Delta$.
- "uniquness" $\left([\omega_0-\omega_1]=0 \in H^2(M^2\setminus\Delta, N(\Delta)\setminus\Delta) \right)$

  $\Rightarrow \exists\, \eta \in \Omega^1(M^2\setminus\Delta, \underline{N(\Delta)\setminus\Delta})$ s.t. $d\eta = \omega_0 - \omega_1$.
- $$\int_{M^2\setminus\Delta}\omega_0^3 - \int_{M^2\setminus\Delta}\omega_1^3 = \int_{M^2\setminus N(\Delta)}\omega_0^3-\omega_1^3 = \int_{M^2\setminus N(\Delta)}\underbrace{(\omega_0-\omega_1)}_{\text{"}d\eta}(\omega_0^2+\omega_0\omega_1+\omega_1^2)$$

$$= \int_{M^2\setminus N(\Delta)} d\left(\eta(\omega_0^2+\omega_0\omega_1+\omega_1^2)\right) \underset{\text{Stokes' thm}}{=} \int_{\partial N(\Delta)} \underbrace{(\eta|_{\partial N(\Delta)})}_{=\,0} \wedge \underbrace{(\varphi_\Delta^*\omega_{S^2}^2|_{\partial N(\Delta)})}_{=\,0} = 0$$

$\leftarrow \dim \omega_{S^2}^2 = 4 \gneq \dim S^2$

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Remark: geometric meaning of ω (propagator)

$\omega \in \Omega^2(M^2 \setminus \Delta)$, propagator

$K_1 \sqcup K_2 \subset M$: 2-comp. link.

$$\left(\begin{array}{l} M : \mathbb{Q}HS^3 \Rightarrow [K_1] = 0 \in H_1(M : \mathbb{Q}) \\ \qquad \Rightarrow {}^\exists n \in \mathbb{N},\ {}^\exists \Sigma^2 \hookrightarrow M \text{ s.t. } \partial\Sigma = nK_1 \text{ (Seifert surf.)} \\ \text{then } lk(K_1, K_2) = \frac{1}{n} \# \Sigma \cap K_2 \end{array} \right)$$

fact

$$\int_{K_1 \times K_2} \omega|_{K_1 \times K_2} = lk(K_1, K_2)$$

Known results about C.S. pert theory at $\mathbb{R}$ (triv. loc. sys.)

- $I_\Theta(M) = \frac{1}{6}\underline{\underline{\lambda^{CW}(M)}} + \frac{3}{4}$(signature defect of the framing)

  Casson-Walker inv.

  (Taubes, ...)

- Θ is a simplest trivalent graph (Jacobi diag.)

  $\rightsquigarrow$ For other (higher degree) graphs Γ,

  $I_\Gamma(M)$ are also defined. (KKT inv.)

- $\{I_\Gamma(M)\}_{\Gamma : \text{Jacobi diag}}$ is universal

  for all finite type invariants

  of $\mathbb{Z}HS^3$s. (Lescop, ...)

3. M : closed 3-mfd. (framed), E : loc. sys ($\neq \mathbb{R}$)

- **Assumption** $H^i(M; E) = 0$, $i \geq 1$.

- $\varphi_\Delta : N(\Delta) \setminus \Delta \longrightarrow S^2$, $(x, y) \longmapsto \dfrac{y - x}{\|y - x\|}$

- $\varphi_\Delta^* \omega_{S^2} \in \Omega^2(N(\Delta) \setminus \Delta)$

$\quad\Big\}$ extend $\downarrow$

$\nexists\, \omega \in \Omega^2(M^2 \setminus \Delta)$, closed

in general

3. M : closed 3-mfd. (framed), E : loc. sys ($\neq \mathbb{R}$)

- Assumption $H^i(M;E) = 0$, $i \geq 1$. (Bott-Cattaneo's inv.)

- $\varphi_\Delta : N(\Delta) \setminus \Delta \longrightarrow S^2$, $(x, y) \longmapsto \frac{y-x}{\|y-x\|}$

- $c(\varphi_\Delta^* \omega_{S^2}) \in \Omega^2(N(\Delta) \setminus \Delta ; E \boxtimes E)$

extend

$$\omega \in \Omega^2(M^2 \setminus \Delta : E \boxtimes E)$$

- $\pi_i : M^2 \to M$, $(x_1, x_2) \mapsto x_i$, projection $(i=1,2)$
- $E \boxtimes E := \pi_1^* E \otimes \pi_2^* E$; a local system on M^2 (or $M^2 \setminus \Delta$)

- Assume $\mathrm{Im}(\underbrace{\pi_1(M, x) \to \mathrm{Aut}\, E_x}_{\text{monodromy of } E}) < SO(E_x)$ $(x \in M)$

$\Rightarrow \exists c : \mathbb{R} \hookrightarrow E \boxtimes E|_{N(\Delta) \setminus \Delta}$

- def $\omega \in \Omega^2(M^2 \setminus \Delta ; E \boxtimes E)$, s.t. closed & $\omega|_{N(\Delta) \setminus \Delta} = c(\varphi_\Delta^* \omega_{S^2})$

propagator

Bott-Cattaneo's construction

- $\omega \in \Omega^2(M^2 \setminus \Delta; E \boxtimes E)$, propagator

- $I_\Theta(M, E) = \underbrace{\int_{M^2 \setminus \Delta}}_{?} \omega^3 \in \Omega^6(M^2 \setminus \Delta; E^{\otimes 3} \boxtimes E^{\otimes 3})$

(Assume E is characterized by
$$\pi_1 M \longrightarrow \underset{\substack{= \\ \text{semi-simple} \\ \text{Lie gp}}}{G} \xrightarrow{Ad} \mathrm{Aut}(\mathfrak{g}) \ (= SO(\mathbb{R}^{\dim \mathfrak{g}}))$$
)

$\Longrightarrow \exists\, tr : E \otimes E \otimes E \to \mathbb{R}$

Killing form

$$(tr(x \otimes y \otimes z) = \langle [x, y], z \rangle, \quad x, y, z \in \mathfrak{g})$$

Lie bracket

Bott-Cattaneo's construction

- $\omega \in \Omega^2(M^2 \setminus \Delta ; E \boxtimes E)$, propagator

def
$$I_\Theta(M,E) = \int_{M^2\setminus\Delta} \underbrace{tr^{\boxtimes 2}\, \omega^3}_{\in\, \Omega^6(M^2\setminus\Delta)} \in \Omega^6(M^2 \setminus \Delta ; \underbrace{E^{\otimes 3}}_{} \boxtimes \underbrace{E^{\otimes 3}}_{})$$

$$E^{\otimes 3} \boxtimes E^{\otimes 3} \xrightarrow{tr \boxtimes tr} \mathbb{R} \boxtimes \mathbb{R} = \mathbb{R}.$$

$^\exists tr : E \otimes E \otimes E \to \mathbb{R}$

Killing form

$$(tr(x \otimes y \otimes z) = \langle [x, y], z \rangle , \quad x, y, z \in \mathfrak{g})$$

Lie bracket

Bott-Cattaneo's construction

"Lem" (Bott-Cattaneo)

① $\exists$ propagator ② propagator is essentially "unique".

"proof" $H^2_{-}(\Delta; E\boxtimes E) = 0 \Rightarrow \exists\, \omega$: propagator

(−1 eigen space of involution $E\boxtimes E \ni x\otimes y \mapsto y \otimes x$)

$H^1_{-}(\Delta; E\boxtimes E) = 0 \Rightarrow$ "uniquness" of propagator.

$\Rightarrow$ "Thm" (Bott-Cattaneo)

$I_\Theta(M,E)$ $\left(= \int_{M^2\setminus\Delta} \mathrm{tr}^{\boxtimes 2}\omega^3\right)$ is an inv. of (M,E).

Bott-Cattaneo's construction

"Lem" (Bott-Cattaneo)

① $\exists$ propagator ② propagator is essentially "unique".

?

"proof" $\cancel{H^2_{-}(\Delta ; E\boxtimes E) = 0} \Longrightarrow {}^{\exists}\omega$: propagator

(-1 eigen space of involution $E\boxtimes E \ni x\otimes y \mapsto y \otimes x$)

$\exists$ error

$\cancel{H^1_{-}(\Delta ; E\boxtimes E) = 0} \Rightarrow$ "uniqueness" of propagator.

$\Longrightarrow$ "Thm" (Bott-Cattaneo)

$I_\Theta(M,E)$ $\left(= \int_{M^2\setminus\Delta} \mathrm{tr}^{\boxtimes 2}\omega^3 \right)$ is an inv. of (M,E).

?

Summary

- M : a closed 3-mfd., E : a local system $\left(\begin{array}{l}\text{corresponding to} \\ \pi_1 M \to G \to \mathrm{Aut}(\mathfrak{g})\end{array}\right)$

- $c(\varphi_\Delta^* \omega_{S^2}) \in \Omega^2(N(\Delta)\setminus\Delta ; E \boxtimes E)$

  $\Big\{$ extend

  $\omega \in \Omega^2(M^2 \setminus \Delta ; E^{\otimes 3} \boxtimes E^{\otimes 3})$, propagator

  s.t. $\begin{cases} \cdot\ d\omega = 0 \\ \cdot\ \omega|_{N(\Delta)\setminus\Delta} = c(\varphi_\Delta^* \omega_{S^2}). \end{cases}$

- $I_\Theta(M, E) = \displaystyle\int_{M^2\setminus\Delta} \mathrm{tr}^{\otimes 2}\, \omega^3 \in \mathbb{R}.$

4. "revised" invariant of (M, E)

- M : a closed 3-mfd., E : a local system (corresponding to $\pi_1 M \to G \to \mathrm{Aut}(\mathfrak{g})$)

- $c(\varphi_\Delta^* \omega_{S^2}) \in \Omega^2(N(\Delta) \setminus \Delta ; E \boxtimes E)$

  $\Big\{$ extend

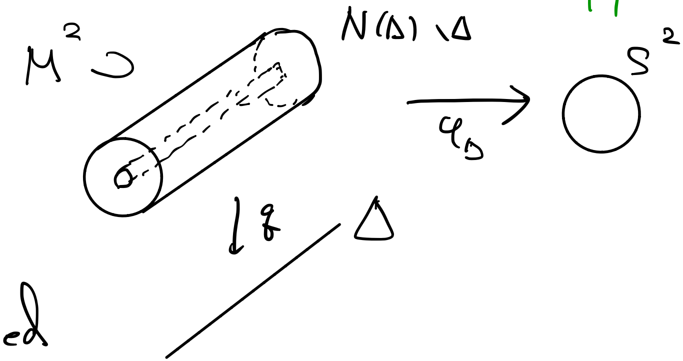
  $\omega \in \Omega^2(M^2 \setminus \Delta ; E^{\otimes 3} \boxtimes E^{\otimes 3})$, propagator

  s.t. $\begin{cases} \cdot\ d\omega = 0 \\ \cdot\ \omega|_{N(\Delta)\setminus\Delta} = c(\varphi_\Delta^* \omega_{S^2}) + q^* \xi \end{cases}$ ("weak" ∂-condi $\Downarrow$ $\exists\, \omega$.)

- $$\underbrace{\int_{M^2 \setminus \Delta} \mathrm{tr}^{\otimes 2}\, \omega^3}_{J_\Theta(\omega)} - \underbrace{3 \int_{M^2 \setminus \Delta} \mathrm{tr}^{\otimes 2} (\pi_1^* \xi\, \omega\, \pi_2^* \xi)}_{J_{\multimap}(\omega, \xi)}$$

"new" ∂-condi. of propagator

$$\begin{array}{ccc} N(\Delta)\setminus\Delta & \xrightarrow{\varphi_\Delta} & S^2 \\ \downarrow{\scriptstyle q\,(\text{proj.})} & & \\ \Delta & & \end{array}$$



Lem $\exists\, \xi \in \Omega^2_-(\Delta : E\boxtimes E)$, closed

s.t. $c(\varphi_\Delta^* \omega_{S^2}) + q^*\xi$ can be extend to $M^2\setminus\Delta$

as a closed 2-form in $\Omega^2(M^2\setminus\Delta : E\boxtimes E)$.

Rmk $[\xi] \in H^2(\Delta : E\boxtimes E)$ is unique.

def $\omega \in \Omega^2(M^2\setminus\Delta : E\boxtimes E)$ is a propagator (w.r.t. ξ)

if $\begin{cases} \cdot\ d\omega = 0 \\ \cdot\ \omega|_{N(\Delta)\setminus\Delta} = c(\varphi_\Delta^*\omega_{S^2}) + q^*\xi \end{cases}$ "new" ∂-condition.

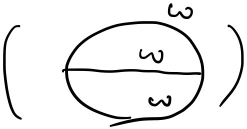
Def of revised invariant

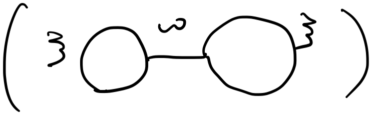
Lem $\exists$ propagator

Rmk ω_0, ω_1 : propagator w.r.t. ξ.

in general, $[\omega_0 - \omega_1] \neq 0 \in H^2(M^2 \setminus \Delta, N(\Delta)\setminus\Delta ; E \otimes E)$.

Def $J_\Theta(\omega) = \int_{M^2\setminus\Delta} \mathrm{tr}^{\otimes 2}\, \omega^3$, $J_{O\text{-}O}(\omega, \xi) = \int_{M^2\setminus\Delta} \mathrm{tr}^{\otimes 2}(\pi_1^*\xi\ \omega\ \pi_2^*\xi)$





Thm. (Cattaneo - S.)

① $J_\Theta(\omega)$, $J_{O\text{-}O}(\omega, \xi)$ are invariants of M and ξ.
(depend on the choice of ξ)

② $I_1(M) := J_\Theta(\omega) - 3J_{O\text{-}O}(\omega, \xi)$ is an inv. of M.

Remark on " $I_\Theta(M) = J_\Theta(\omega) - 3 J_{\circ\!-\!\circ}(\omega, \xi)$ "

- Θ, $\circ\!-\!\circ$; degree 1 (Feynman) diagram.

- perturbative expansion $\rightsquigarrow$ $J_\Theta(\omega)$, $J_{\circ\!-\!\circ}(\omega, \xi)$

(cf. Axelrod-Singer's Chern-Simons pert. theory.)

- When $G = SU(2)$, $[\xi] = 0 \in H^2(\Delta ; E \boxtimes E)$.

  $\to$ we can take $\xi = 0$

  $\to$ $J_{\circ\!-\!\circ}(\omega, \xi) = 0$, $\omega|_{N(\Delta)\setminus\Delta} = c(\varphi_\Delta^* \omega_{S^2}) + \cancel{\varphi^* \xi}$

Proof of well-def. of $I_1(M)$ ①

[① $J_\Theta(\omega)$, $J_{\Theta\text{-}\Theta}(\omega, \xi)$ are invariants of M and ξ.]

proof $\omega_0, \omega_1 \in \Omega^2(M^2 \setminus \Delta ; E \boxtimes E)$, propagator

$$\Rightarrow \exists \eta \in \Omega^1(M^2 \setminus \Delta ; E \boxtimes E) \text{ s.t. } d\eta = \omega_1 - \omega_0$$

$$\left(\notin \Omega^1(M^2 \setminus \Delta, \underline{\underline{N(\Delta) \setminus \Delta}} ; E \boxtimes E) \right)$$

Proof of well-def. of $I_1(M)$ ①

[① $J_\Theta(\omega)$, $J_{\Theta_0}(\omega, \xi)$ are invariants of M and ξ.]

<u>proof</u> $\omega_0, \omega_1 \in \Omega^2(M^2 \setminus \Delta; E \boxtimes E)$, propagator

$\Rightarrow \exists \eta \in \Omega^1(M^2 \setminus \Delta; E \boxtimes E)$ s.t. $d\eta = \omega_1 - \omega_0$

$$J_\Theta(\omega_0) - J_\Theta(\omega_1) = \int_{M^2 \setminus \Delta} \mathrm{tr}^{\otimes 2}(\omega_0^3 - \omega_1^3) \underset{\substack{\uparrow \\ \omega_0 = \omega_1 \text{ on } N(\Delta) \setminus \Delta}}{=} \int_{M^2 \setminus N(\Delta)} \mathrm{tr}^{\otimes 2}\big(\underbrace{(\omega_0 - \omega_1)}_{= d\eta}(\omega_0^2 + \omega_0\omega_1 + \omega_1^2)\big)$$

$$= \int_{\partial N(\Delta)} \mathrm{tr}^{\otimes 2}\left(\eta\big|_{\partial N(\Delta)} (c(\varphi_\Delta^* \omega_{S^2}) + q^* \xi)^2\right)$$

$$= \int_{\partial N(\Delta)} \mathrm{tr}^{\otimes 2}\left(\eta\big|_{\partial N(\Delta)} \left(\underbrace{c(\varphi_\Delta^* \omega_{S^2}^2)}_{= 0} + c(\varphi_\Delta^* \omega_{S^2})\, q^* \xi + \underbrace{q^* \xi^2}_{= 0}\right)\right)$$

dim. reason

$$= \int_{\partial N(\Delta)} \mathrm{tr}^{\otimes 2}\left(\eta\big|_{\partial N(\Delta)} c(\varphi_\Delta^* \omega_{S^2})\, q^* \xi\right) \underset{\uparrow}{=} 0 \;!!$$

($H^*(\Delta : E) = 0$ & a property of $\mathrm{tr}^{\otimes 2}$)

Proof of well-def. of $I_1(M)$ ②

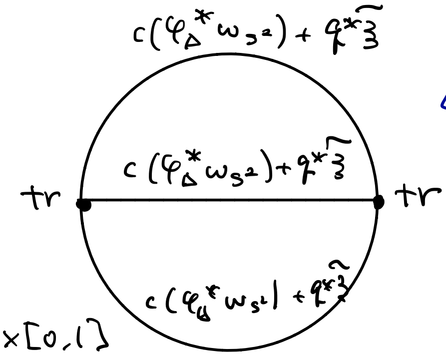
[② $I_1(M) := J_\theta(\omega) - 3J_{o\text{-}o}(\omega, \xi)$ is an inv. of M.]

proof (idea)

- $\xi' \in \Omega^2(\Delta ; E \boxtimes E)$, alt. choice. ($\omega'$: propagator w.r.t. ξ')

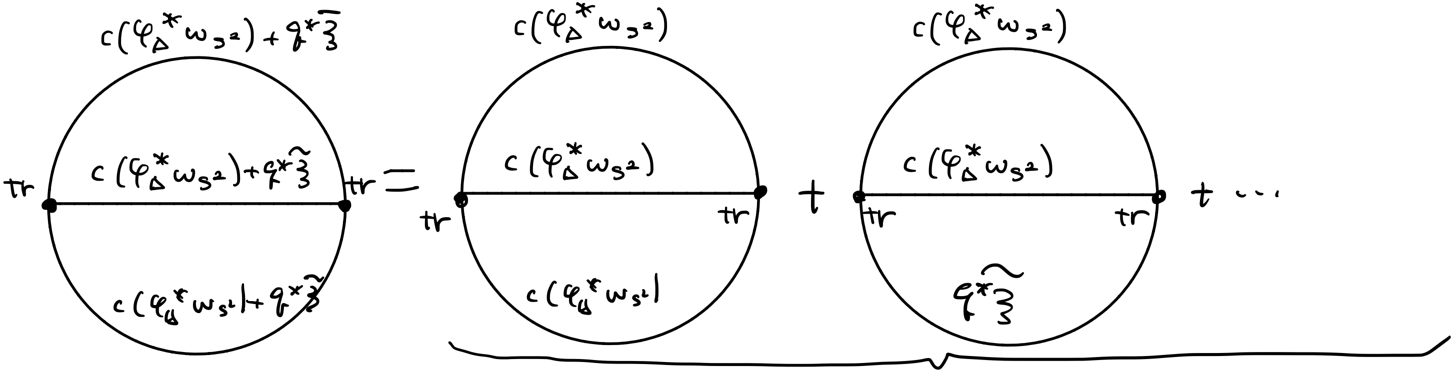
$\Rightarrow \exists\, \tilde{\xi} \in \Omega^2(\Delta \times [0,1] ; E \boxtimes E)$, closed, $\tilde{\xi}|_{\Delta \times 0} = \xi$, $\tilde{\xi}|_{\Delta \times 1} = \xi'$.

$$\to J_\theta(\omega) - J_\theta(\omega') = \cdots = \int_{(N(\Delta)\setminus\Delta)\times[0,1]} \mathrm{tr}^2\left(c(\varphi_\Delta^* \omega_{S^2}) + q^* \tilde{\xi}\right)^3$$

$$\int_{(N(\Delta)\setminus\Delta)\times[0,1]}$$ 

↶ graphical image

Proof of well-def. of $I_1(M)$ (2)

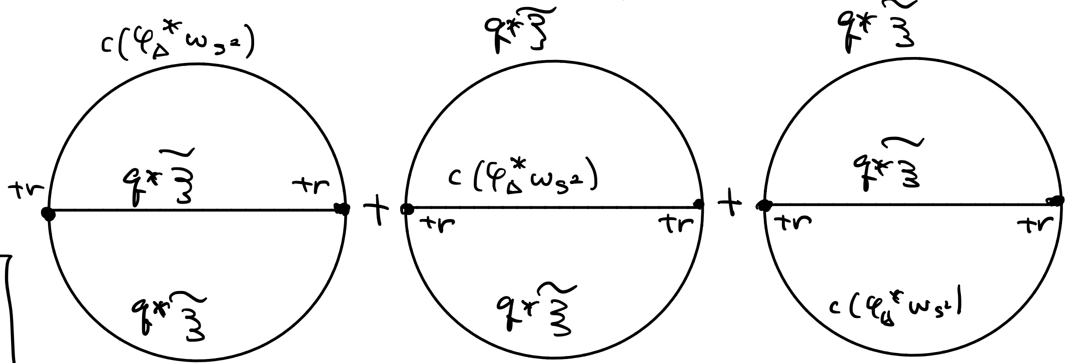


(8 terms)

=

$$\begin{bmatrix} (\varphi_\Delta^* \omega_{S^2})^2 = 0 \\ \tilde{\xi}^3 = 0 \end{bmatrix}$$

dim. reason

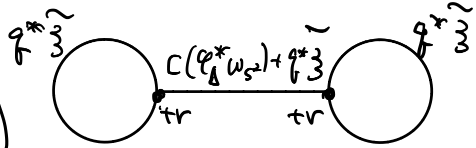


Proof of well-def. of $I_1(M)$ ②

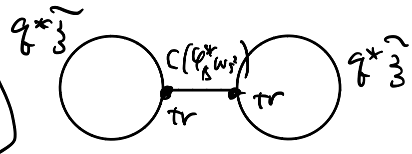
$$\Rightarrow J_\Theta(\omega) - J_\Theta(\omega') = 3\int_{(N(\Delta)\setminus\Delta)\times[0,1]} \mathrm{tr}^{\otimes 2}\left(c(\varphi_\Delta^*\omega_{S^2})\, q^*\tilde{\xi}^{2}\right)$$

Similarly, $J_{\circ\!-\!\circ}(\omega,\xi) - J_{\circ\!-\!\circ}(\omega',\xi')$

$$= \cdots = \int_{(N(\Delta)\setminus\Delta)\times[0,1]}$$



$$= \int_{(N(\Delta)\setminus\Delta)\times[0,1]}$$



$q^*\tilde{\xi}^{3} = 0$

$$\Rightarrow J_{\circ\!-\!\circ}(\omega,\xi) - J_{\circ\!-\!\circ}(\omega',\xi') = \int_{(N(\Delta)\setminus\Delta)\times[0,1]} \mathrm{tr}^{\otimes 2}\left(c(\varphi_\Delta^*\omega_{S^2})\cdot q^*\tilde{\xi}^{2}\right)$$

therefore, $J_\Theta(\omega) - 3J_{\circ\!-\!\circ}(\omega,\xi) = J_\Theta(\omega') - 3J_{\circ\!-\!\circ}(\omega',\xi')$ //

Remark (framing correction)

- M : a closed 3-mfd, $\tau_0, \tau_1 : TM \xrightarrow{\cong} M \times \mathbb{R}^3$, framing

  ($\rightarrow (M, \tau_0), (M, \tau_1)$: framed 3-mfd.)

- E : a loc. sys. on M characterized by $\pi_1 M \to G \to \mathrm{Aut}(\mathfrak{g})$

$\rightsquigarrow \quad I_1((M,\tau_0),E), \quad I_1((M,\tau_1),E)$

Prop (Cattaneo-S.)

$$\boxed{I_1((M,\tau_0),E) - I_1((M,\tau_1),E) = \frac{3}{4}(\dim \mathfrak{g})^2 \left(\delta(\tau_0) - \delta(\tau_1)\right)}$$

the signature defect of τ_0, τ_1.

Cor.

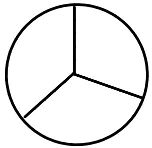
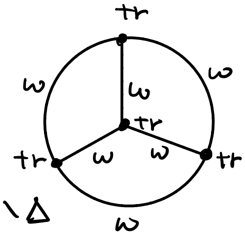
$$I_1((M,\tau_0),E) - \frac{3}{4}(\dim \mathfrak{g})^2 \delta(\tau_0)$$

is an inv. of (M, E).

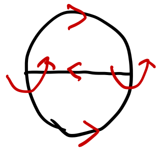
5. higher degree inv. (review) $G = SU(2)$

(→ we take $\xi = 0$)

- Θ, (O–O) … degree 1 graph.
- $\forall \Gamma$ … trivalent graph (w/o —○), $J_\Gamma(\omega)$ is defined

e.g. $\Gamma =$  $\rightsquigarrow J_\Gamma(\omega)$ "=" $\displaystyle\int_{M^4 \setminus \Delta}$ 

- $\mathcal{A} := \left(\mathbb{Q}\text{-vect. space. spaned by all oriented trivalent graph} \right) / \sim$

e.g. 

$\circlearrowleft \delta$: differential.

- For $\sum_i a_i \Gamma_i \in \mathcal{A}$, s.t. $\delta(\sum_i a_i \Gamma_i) = 0$,

$$\sum_i a_i J_{\Gamma_i}(\omega) \quad \text{is an inv. of } (M, E).$$