Divisibility of Lee's class and its relation with Rasmussen's invariant

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Introduction : History

- [Kho00] M. Khovanov, “A categorification of the Jones polynomial”.
  - **Khovanov homology** - a bigraded link homology theory, constructed combinatorially from a planar link diagram.
  - Its graded Euler characteristic gives the Jones polynomial.
- [Lee05] E. S. Lee, “An endomorphism of the Khovanov invariant”.
  - **Lee homology** - a variant of Khovanov homology, originally introduced to prove the “Kight move conjecture” for the $\mathbb{Q}$-Khovanov homology of alternating knots.
- [Ras10] J. Rasmussen, “Khovanov homology and the slice genus”.
  - s**-invariant** - an integer valued knot invariant obtained from Lee homology.
  - s gives a combinatorial proof for the Milnor conjecture.

Introduction : Khovanov homology

Khovanov homology H_{Kh} is a bigraded link homology theory, constructed combinatorially from a planar link diagram.

Theorem ([Kho00, Theorem 1])

For any diagram D of an oriented link L, the isomorphism class of $H_{Kh}^{\cdot\cdot}(D; R)$ (as a bigraded R-module) is an invariant of L.

Proposition ([Kho00, Proposition 9])

The graded Euler characteristic of $H_{Kh}^{\cdot\cdot}(L; \mathbb{Q})$ gives the (unnormalized) Jones polynomial of L:

$$\sum_{i,j} (-1)^i q^j \dim_{\mathbb{Q}}(H_{Kh}^{ij}(L; \mathbb{Q})) \;=\; (q + q^{-1}) V(L)|_{\sqrt{t}=-q}.$$

Introduction : Khovanov homology

0_1

j \ i	0
1	1
-1	1

3_1

j \ i	-3	-2	-1	0
-1				1
-3				1
-5		1		
-7				
-9	1			

4_1

j \ i	-2	-1	0	1	2
5					1
3					
1			1	1	
-1		1	1		
-3					
-5	1				

Figure 1: $H_{Kh}(K;\mathbb{Q})$ for $K = 0_1, 3_1, 4_1$

Introduction : Khovanov homology

9_{42}

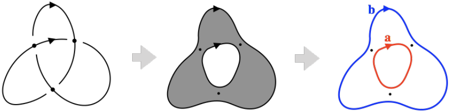
	-4	-3	-2	-1	0	1	2
7							1
5							
3					1	1	
1				1	1		
-1				1	1		
-3		1	1				
-5							
-7	1						

Figure 2: $H_{Kh}(K;\mathbb{Q})$ for $K = 9_{42}$

Introduction : Lee homology and Lee's classes

Lee homology H_{Lee} is a variant of Khovanov homology. Although the construction is similar, when $R = \mathbb{Q}$ the structure of H_{Lee} is strikingly simple.

For a knot diagram D, there are two distinct classes $[\alpha], [\beta]$ constructed combinatorially from D.



Theorem ([Lee05, Theorem 4.2])

When $R = \mathbb{Q}$, the two classes $[\alpha], [\beta]$ form a basis of $H_{Lee}(D; \mathbb{Q})$.

Remark

For a link diagram D with ℓ components, there are 2^ℓ distinct classes $[\alpha(D, o)]$ one for each alternative orientation o of D. These form a basis of $H_{Lee}(D; \mathbb{Q})$.

Introduction : Rasmussen's s-invariant

Rasmussen introduced in [Ras10] an integer-valued knot invariant, called the s-**invariant**, based on $\mathbb{Q}$-Lee homology.

H_{Lee} is not bigraded (unlike H_{Kh}), but admits a filtration by q-degree. For a knot K, the s-invariant is defined by

$$s(K) := \frac{q_{\max} + q_{\min}}{2},$$

where $q_{\max}$ (resp. $q_{\min}$) denotes the maximum (resp. minimum) q-degree of $H_{Lee}(K; \mathbb{Q})$.

Rasmussen showed that $[\alpha], [\beta]$ are invariant (up to unit) under the Reidemeister moves. Hence Rasmussen called them the "**canonical generators**" of $H_{Lee}(K; \mathbb{Q})$.

This fact is used to prove the important properties of s.

Introduction : Properties of s

Theorem ([Ras10, Theorem 2])

s defines a homomorphism from the knot concordance group in S^3 to $2\mathbb{Z}$:

$$s\colon Conc(S^3) \to 2\mathbb{Z}.$$

Theorem ([Ras10, Theorem 1])

s gives a lower bound of the slice genus:

$$|s(K)| \leq 2g_*(K).$$

Theorem ([Ras10, Theorem 4])

If K is a positive knot, then

$$s(K) = 2g_*(K) = 2g(K).$$

Introduction : Properties of s

With the above three properties of s, one obtains:

Corollary (The Milnor Conjecture, [Mil68])

The (smooth) slice genus and the unknotting number of the (p, q) torus knot are both equal to $(p-1)(q-1)/2$.

Remark

The Milnor Conjecture was first proved by Kronheimer and Mrowka in [KM93] using gauge theory, but Rasmussen's result was notable since it provided a purely combinatorial proof.

Our observations

Now we consider Lee homology over $\mathbb{Z}$. Let D be a knot diagram, and denote

$$H_{Lee}(D;\mathbb{Z})_f = H_{Lee}(D;\mathbb{Z})/Tor.$$

The two classes $[\alpha],[\beta]$ can be defined over $\mathbb{Z}$, but they do not form a basis of $H_{Lee}(D;\mathbb{Z})_f \cong \mathbb{Z}^2$.

We created a computer program[1] that calculates the components of $[\alpha],[\beta]$ with respect to some basis of $H_{Lee}(D;\mathbb{Z})_f$. It turned out, that for any prime knot diagram of crossing number up to 11, only **2-powers** appear in those components.

Question

Where does the 2-powers come from, and what information can we extract from the 2-divisibility of $[\alpha],[\beta]$*?*

[1] `https://github.com/taketo1024/SwiftyMath`

Computational results

```
3_1
b⊛a(111) = [2, -2]
a⊛b(111) = [-2, -2]

4_1
a⊛b⊛a(0011) = [-2, -2]
b⊛a⊛b(0011) = [2, -2]

5_1
b⊛a(11111) = [2, -2]
a⊛b(11111) = [-2, -2]

5_2
b⊛a⊛b⊛a(11111) = [-8, 8]
a⊛b⊛a⊛b(11111) = [-8, -8]

6_1
a⊛b⊛a⊛b⊛a(110011) = [8, 8]
b⊛a⊛b⊛a⊛b(110011) = [8, -8]

6_2
a⊛b⊛a(110011) = [2, 2]
b⊛a⊛b(110011) = [2, -2]

6_3
a⊛b⊛b(001110) = [-2, 2]
b⊛a⊛a(001110) = [2, 2]

7_1
b⊛a(1111111) = [2, -2]
a⊛b(1111111) = [-2, -2]

7_2
b⊛a⊛b⊛a⊛b⊛a(1111111) = [-32, 32]
a⊛b⊛a⊛b⊛a⊛b(1111111) = [32, 32]

7_3
b⊛a⊛b⊛a(0000000) = [-1, 1]
a⊛b⊛a⊛b(0000000) = [1, 1]

7_4
b⊛a⊛b⊛a⊛b⊛a(0000000) = [-1, 1]
a⊛b⊛a⊛b⊛a⊛b(0000000) = [1, 1]

7_5
b⊛a⊛b⊛a(1111111) = [-8, 8]
a⊛b⊛a⊛b(1111111) = [-8, -8]

7_6
a⊛b⊛b⊛b(1110011) = [4, -4]
b⊛a⊛a⊛a(1110011) = [4, 4]

7_7
a⊛b⊛a⊛a(1100100) = [2, -2]
b⊛a⊛b⊛b(1100100) = [-2, -2]
```

Figure 3: Computational results[2]

[2] https://git.io/fphro

Overview of our results (1/2)

We consider the question in a more generalized setting. There exists a family of Khovanov-type homology theories $\{H_c(-; R)\}_{c \in R}$ over a commutative ring R parameterized by $c \in R$.

For each $c \in R$, Lee's classes $[\alpha], [\beta]$ of a knot diagram D can also be defined in $H_c(D; R)$.

If R is an integral domain and c is non-zero, non-invertible, then we can define the c-**divisibility** of $[\alpha]$ (modulo torsions) by the exponent of its c-power factor. We denote it by $k_c(D)$.

By inspecting the variance of k_c under the Reidemeister moves, we prove that

$$\bar{s}_c(K) := 2k_c(D) + w(D) - r(D) + 1$$

is a knot invariant, where w is the writhe, and r is the number of Seifert circles.

Overview of our results (2/2)

Again by computations, we saw that values of $\bar{s}_2(-;\mathbb{Z})$ coincide with values of s for all prime knots of crossing number up to 11.

Question

Are all $\bar{s}_c$ equal to s?

The following theorems support the affirmative answer.

Theorem (S.)

Each $\bar{s}_c$ possesses properties common to s. In particular, the each $\bar{s}_c$ can be used to reprove the Milnor conjecture.

Theorem (S.)

If $(R, c) = (\mathbb{Q}[h], h)$, then the knot invariant $\bar{s}_h$ coincides with s

$$s(K) = \bar{s}_h(K; \mathbb{Q}[h]).$$

Conventions

In this talk, all knots and links are assumed to be **oriented**. For simplicity, we mainly focus on **knots**, but many of the results can be generalized to links.

Contents

Frobenius algebra (1/2)

Let R be a commutative ring with unity. A **Frobenius algebra** over R is a quintuple $(A, m, \iota, \Delta, \varepsilon)$ satisfying:

1. (A, m, ι) is an associative R-algebra with multiplication $m : A \otimes A \to A$ and unit $\iota : R \to A$,
2. (A, Δ, ε) is a coassociative R-coalgebra with comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to R$, and
3. the Frobenius relation holds:

$$\Delta \circ m = (id \otimes m) \circ (\Delta \otimes id) = (m \otimes id) \circ (id \otimes \Delta).$$

Frobenius algebra (2/2)

A commutative Frobenius algebra A gives a 1+1 TQFT

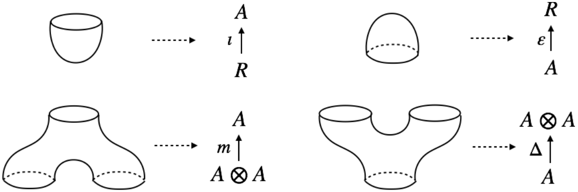
$$\mathcal{F}_A : Cob_2 \longrightarrow Mod_R,$$

by mapping:

- Objects:

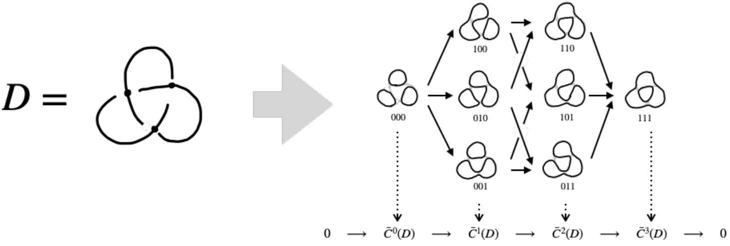
$$\underbrace{\bigcirc \sqcup \cdots \sqcup \bigcirc}_{r} \longrightarrow \underbrace{A \otimes \cdots \otimes A}_{r}$$

- Morphisms:



Construction of the chain complex

Let D be a link diagram with n crossings. The 2^n resolutions of the crossings yields a commutative cubic diagram in Cob_2.



By applying $\mathcal{F}_A$ we obtain a commutative cubic diagram in Mod_R.

Then we turn this cube skew commutative by appropriately adjusting the signs of the edge maps.

Finally we fold the cube and obtain a chain complex $C_A(D)$ and its homology $H_A(D)$.

Khovanov homology and its variants

Khovanov's original theory is given by $A = R[X]/(X^2)$. Other variant theories are given by:

- $A = R[X]/(X^2 - 1) \quad \rightarrow$ Lee's theory
- $A = R[X]/(X^2 - hX) \rightarrow$ Bar-Natan's theory

Khovanov unified these theories in [Kho06] by considering the following special Frobenius algebra with $h, t \in R$:

$$A_{h,t} = R[X]/(X^2 - hX - t).$$

Denote the corresponding chain complex by $C_{h,t}(D; R)$ and its homology by $H_{h,t}(D; R)$. The isomorphism class of $H_{h,t}(D; R)$ is invariant under Reidemeister moves, thus gives a link invariant.

Contents

Generalizing Lee's classes (1/2)

In order to generalize Lee's classes $[\alpha], [\beta]$ in $H_{h,t}(D; R)$, we assume (R, h, t) satisfies the following condition:

Condition

$X^2 - hX - t$ factors into linear polynomials in $R[X]$.

This is equivalent to:

Condition

There exists $c \in R$ such that $h^2 + 4t = c^2$ and $(h \pm c)/2 \in R$.

With this condition, fix one square root $c = \sqrt{h^2 + 4t}$, and let $X^2 - hX - t = (X - u)(X - v)$ with $c = v - u$. Define

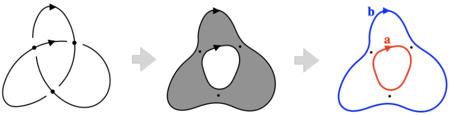
$$\mathbf{a} = X - u, \quad \mathbf{b} = X - v \ \in \ A.$$

Generalizing Lee's classes (2/2)

With **a** and **b**, the multiplication and comultiplication on A diagonalizes as:

$$
\begin{aligned}
m(\mathbf{a}\otimes\mathbf{a}) &= c\mathbf{a}, & \Delta(\mathbf{a}) &= \mathbf{a}\otimes\mathbf{a},\\
m(\mathbf{a}\otimes\mathbf{b}) &= 0, & \Delta(\mathbf{b}) &= \mathbf{b}\otimes\mathbf{b}\\
m(\mathbf{b}\otimes\mathbf{a}) &= 0 & &\\
m(\mathbf{b}\otimes\mathbf{b}) &= -c\mathbf{b} & &
\end{aligned}
$$

We define the cycles $\alpha, \beta \in C_{h,t}(D; R)$ by the orientation preserving resolution of D.



Remark

For a link diagram D with ℓ components, there are 2^ℓ distinct cycles $\alpha(D, o)$ one for each alternative orientation o of D.

Reduction of parameters

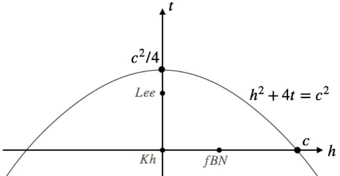
In our setting, we may reduce the parameters (h, t) to a single parameter c.

Proposition

For another (h', t') such that $c = \sqrt{h'^2 + 4t'}$, the corresponding groups $H_{h,t}(D; R)$ and $H_{h',t'}(D; R)$ are naturally isomorphic, and under the isomorphism the Lee classes correspond one-to-one. □

Thus we denote the isomorphism class by $H_c(D; R)$ and regard $[\alpha], [\beta] \in H_c(D; R)$.

The figure depicts the (h, t)-parameter space, where each point (h, t) corresponds to $H_{h,t}(D; R)$ and the parabola $h^2 + 4t = c^2$ corresponds to the isomorphism class $H_c(D; R)$.



Generalizing Lee's theorem

The following proposition generalizes Lee's theorem for $\mathbb{Q}$-Lee homology ($\mathbb{Q}$-Lee homology corresponds to $(R, c) = (\mathbb{Q}, 2)$).

Proposition

If c is invertible in R, then $\{[\alpha], [\beta]\}$ *form a basis of* $H_c(D; R)$.

Proof.

A is free over R with basis $\{1, X\}$. Now $\{\mathbf{a}, \mathbf{b}\}$ also form a basis of A, since the transformation matrix $\begin{pmatrix} -u & -v \\ 1 & 1 \end{pmatrix}$ has determinant $v - u = c$.

By the *admissible colorings decomposition* of $C_c(D; R)$ (proposed by Wehrli in [Weh08]), one can show that the subcomplex generated by α and β becomes $H_c(D; R)$, whereas the remaining part is acyclic. □

Remark

For a link diagram D, the 2^{ℓ} classes $\{[\alpha(D, o)]\}_o$ form a basis of $H_c(D; R)$.

Correspondence under Reidemeister moves (1/2)

Next, the following proposition generalizes the "invariance of $[\alpha]$ and $[\beta]$ (up to unit) in $\mathbb{Q}$-Lee theory".

Proposition

Suppose D, D' are two diagrams related by a single Reidemeister move. Under the corresponding isomorphism:

$$\rho : H_c(D; R) \to H_c(D'; R)$$

there exists some $j \in \{0, \pm 1\}$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ such that the α, β-classes of D and D' are related as:

$$[\alpha'] = \varepsilon c^j \cdot \rho[\alpha],$$
$$[\beta'] = \varepsilon' c^j \cdot \rho[\beta].$$

(Here c is not necessarily invertible, so when $j < 0$ the equation $z = c^j w$ is to be understood as $c^{-j} z = w$.)

Correspondence under Reidemeister moves (2/2)

Proposition (continued)

Moreover the exponent j is given by

$$j = \frac{\Delta r - \Delta w}{2}$$

where r denotes the number of Seifert circles, w denotes the writhe, and the prefixed Δ is the difference of the corresponding numbers for D and D'.

Proof.

The isomorphism ρ is given explicitly, and the proof is done by checking all possible patterns of $[\alpha]$ and those images under ρ. □

Note that $[\alpha], [\beta]$ are invariant (up to unit) iff c is invertible.

Remark

Similar statement holds for link diagrams.

Summary

- We defined a family of Khovanov-type link homology theories $\{H_c(-; R)\}_{c \in R}$, where Knovanov's theory corresponds to $c = 0$ and Lee's theory corresponds to $c = 2$.

- For each $c \in R$, we generalized Lee's classes $[\alpha], [\beta]$ of a knot diagram D in $H_c(D; R)$.

- $[\alpha], [\beta]$ form a basis of $H_c(D; R)$ and are invariant (up to unit) under the Reidemeister moves iff c is invertible.

Thus the situation is completely analogous to $\mathbb{Q}$-Lee theory when c is invertible. Our main concern is when c is not invertible.

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c-divisibility of the α-class

Let R be an integral domain, and $c \in R$ be a non-zero non-invertible element in R. Denote

$$H_c(D;R)_f = H_c(D;R)/Tor.$$

By abuse of notation, we denote the images of $[\alpha], [\beta]$ in $H_c(D;R)_f$ by the same symbols.

Definition

For any knot diagram D, define the c-**divisibility** of $[\alpha]$ by:

$$k_c(D) := \max\{k \geq 0 \mid [\alpha] \in c^k H_c(D;R)_f \}.$$

Note that there is a filtration:

$$H_c(D;R)_f \supset cH_c(D;R)_f \supset \cdots \supset c^k H_c(D;R)_f \supset \cdots$$

so $k_c(D)$ is the maximal filtration level that contains $[\alpha]$.

Basic properties of $k_c(D)$

Proposition

$$0 \leq k_c(D) \leq n^-(D)$$

.

In particular if D is positive, then $k_c(D) = 0$. We can regard k_c as the measure of the "non-positivity" of the diagram.

Proposition

1. $k_c(D) = k_c(-D)$.
2. $k_c(D) + k_c(D') \leq k_c(D \sqcup D')$.
3. $k_c(D \# D') \leq k_c(D \sqcup D') \leq k_c(D \# D') + 1$.

Variance of k_c under Reidemeister moves

Proposition

Let D, D' be two diagrams of the same knot. Then

$$\Delta k_c = \frac{\Delta r - \Delta w}{2},$$

where the prefixed Δ is the difference of the corresponding numbers for D and D'. □

Theorem (S.)

For any knot K,

$$\bar{s}_c(K) := 2k_c(D) - r(D) + w(D) + 1$$

is an invariant of K. □

Remark

$\bar{s}_c$ can also be defined for links.

Basic properties of $\bar{s}_c(K)$

Proposition

$$\bar{s}_c(K) \in 2\mathbb{Z}.$$

Proposition

1. $\bar{s}_c(L) = \bar{s}_c(-L)$.
2. $\bar{s}_c(L \sqcup L') \geq \bar{s}_c(L) + \bar{s}_c(L') - 1$.
3. $\bar{s}_c(L \# L') = \bar{s}_c(L \sqcup L') \pm 1$.

Contents

Behavior under cobordisms (1/2)

The important properties of s are obtained by inspecting its behavior under cobordisms between knots. By tracing the arguments given in [Ras10], we obtain a similar proposition for $\bar{s}_c$.

Proposition (S.)

If S is an oriented connected cobordism between knots K, K', then

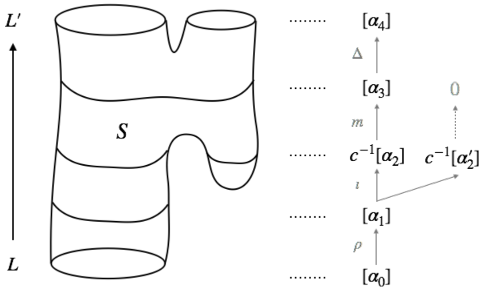
$$|\bar{s}_c(K') - \bar{s}_c(K)| \leq -\chi(S).$$

Remark

Similar statement holds for links.

Behaviour under cobordisms (2/2)

Proof sketch.



Decompose S into elementary cobordisms such that each factor corresponds to a Reidemeister move or a Morse move. Inspect the successive images of the α-class at each level. □

Consequences

The previous proposition implies properties of $\bar{s}_c$ that are common to the s-invariant:

Theorem (S.)

- *$\bar{s}_c$ is a knot concordance invariant in S^3.*
- *For any knot K,*
$$|\bar{s}_c(K)| \leq 2g_*(K).$$
- *If K is a positive knot, then*
$$\bar{s}_c(K) = 2g_*(K) = 2g(K).$$

These properties suffice to reprove the Milnor conjecture.

Contents

The refined canonical generators (1/2)

Now we focus on the case

$$(R, c) = (\mathbb{Q}[h], h),\ \deg h = -2$$

and prove that $\bar{s}_h(-; \mathbb{Q}[h])$ coincides with s.

Recall that in general $[\alpha], [\beta]$ do not form a basis of $H_c(D; R)_f$. However in the above case, we can "normalize" them to obtain a class $[\zeta]$ such that $\{[\zeta], X[\zeta]\}$ is a basis of $H_c(D; R)_f$.

Moreover they are invariant under the Reidemeister moves, so it is reasonable to call them the "**canonical generators**" of $H_c(D; R)_f$.

Remark

X denotes an action on $H_c(D; R)$ defined by merging a circled labeled X to a neighborhood of a fixed point of D.

The refined canonical generators (2/2)

Proposition

There is a unique class $[\zeta] \in H_h(D; R)_f$ *such that*

- $[\zeta], X[\zeta]$ *form a basis of* $H_h(D; R)_f$*, and are invariant under the Reidemeister moves.*
- $[\alpha], [\beta]$ *can be described as*

$$\begin{aligned} [\alpha] &= \quad h^k(\quad (h/2)[\zeta] + X[\zeta]) \\ [\beta] &= (-h)^k(-(h/2)[\zeta] + X[\zeta]), \end{aligned}$$

where $k = k_h(D)$. □

Remark (1)

Unlike $[\alpha]$ or $[\beta]$, the definition of $[\zeta]$ is non-constructive.

Remark (2)

Currently this result is only obtained for knots.

The homomorphism property of $\bar{s}_h$

From the description of $[\alpha], [\beta]$ by the class $[\zeta]$, we can prove:

Proposition

For $(R, c) = (\mathbb{Q}[h], h)$,

- $k_h(D) + k_h(\overline{D}) = r(D) - 1$.
- $k_h(D \# D') = k_h(D) + k_h(D')$.

where $\overline{D}$ *denotes the mirror image of* D.

And we obtain:

Theorem (S.)

For $(R, c) = (\mathbb{Q}[h], h)$, *the invariant* $\bar{s}_h$ *defines a homomorphism*

$$\bar{s}_h \colon Conc(S^3) \to 2\mathbb{Z}.$$

Coincidence with the s-invariant (1/4)

Theorem (S.)

For $(R, c) = (\mathbb{Q}[h], h)$, *our* $\bar{s}_h(-; \mathbb{Q}[h])$ *coincides with* s*:*

$$s(K) = 2k_h(D) + w(D) - r(D) + 1.$$

Remark (1)

There is a well known lower bound for s [Shu07, Lemma 1.3]

$$s(K) \geq w(D) - r(D) + 1,$$

so $2k_h(D)$ gives the correction term of the inequality.

Remark (2)

Currently this result is only obtained for knots.

Coincidence with the s-invariant (2/4)

Proof.

Since both s and $\bar{s}_h$ changes sign by mirroring, it suffices to prove

$$s(K) \geq \bar{s}_h(K).$$

Recall that $s(K)$ is defined by the (filtered) q-degree of $H_2(D; \mathbb{Q})$. On the other hand, q-degree on $H_h(D; \mathbb{Q}[h])$ gives a strict grading. There is a q-degree non-decreasing map

$$\pi : H_h(D; \mathbb{Q}[h]) \to H_2(D; \mathbb{Q})$$

induced from $\mathbb{Q}[h] \to \mathbb{Q}, \ h \mapsto 2$.

Denote by $[\alpha_2], [\alpha_h]$ the α-classes of D in $H_2(D; F)$, $H_h(D; \mathbb{Q}[h])$ respectively. Then by definition $\pi[\alpha_h] = [\alpha_2]$.

Coincidence with the s-invariant (3/4)

Proof continued.

Let $[\alpha_h] = h^k[\alpha'_h]$ with $k = k_h(D)$. Then from $\deg h = -2$,

$$\mathrm{qdeg}_h([\alpha'_h]) = \mathrm{qdeg}_h([\alpha_h]) + 2k = w(D) - r(D) + 2k,$$

so we have

$$\begin{aligned} s(K) &= \mathrm{qdeg}([\alpha_2]) + 1 \\ &= \mathrm{qdeg}(\pi[\alpha_h]) + 1 \\ &= \mathrm{qdeg}(\pi[\alpha'_h]) + 1 \\ &\geq \mathrm{qdeg}_h([\alpha'_h]) + 1 \\ &= w(D) - r(D) + 2k + 1 \\ &= \bar{s}_h(K; F[h]). \end{aligned}$$

□

Coincidence with the s-invariant (4/4)

Corollary

$$s(K) = \mathrm{qdeg}_h[\zeta] - 1.$$

Remark

Khovanov gave an alternative definition of s in [Kho06] by the q-degree of the generator of $H_{0,t}(D; \mathbb{Q}[t])$ where $\deg t = -4$. The equivalence of the two definitions can be proved using the above result.

Contents

Question (1)

Are all $\bar{s}_c$ equal to s?

Remark (1)

The s-invariant can be defined over any field F of char $F \neq 2$. In fact we can prove that

$$s(-; F) = \bar{s}_h(-; F[h]).$$

It is an open question whether $s(-; F)$ for char $F \neq 2$ are all equal or not [LS14, Question 6.1]. If [Question 1] is solved affirmatively, then it follows that $s(-; F)$ are all equal.

Remark (2)

In [LS14], an alternative definition of s over any field F (including char $F = 2$) is given. It is defined similarly to the original one, but is based on the filtered Bar-Natan homology. However C.Seed showed by direct computation that $K = K14n19265$ has $s(K; \mathbb{Q}) = 0$ but $s(K; \mathbb{F}_2) = -2$.

Question (2)

Can we construct $[\zeta] \in H_c(D; R)$ *for any* (R, c)*?*

The existence of $[\zeta] \in H_h(D; \mathbb{Q}[h])$ was the key to prove $s = \bar{s}_h$.

If such class exists in general, then we can expect that Question (1) can also be solved. However the current proof for $(R, c) = (\mathbb{Q}[h], h)$ cannot be applied to the general case.

Maybe we can find a more geometric (or combinatorial) construction.

Question (3)

Does $s = \bar{s}_c(-;\mathbb{Q}[h])$ *also hold for links?*

The definition of s for links is given by Beliakova and Wehrli in [BW08]. Our $\bar{s}_c$ can also be defined for links, so the question makes sense.

Maybe we can construct the canonical generators $[\zeta_1], \cdots, [\zeta_{2^\ell}]$ of $H_h(D;\mathbb{Q}[h])$ such that the α-classes $\{[\alpha(D,o)]\}_o$ can be described by them.

Thank you!

https://arxiv.org/abs/1812.10258

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