Finiteness of the image of the Reidemeister torsion of a splice

$\tau_\rho(M^3) \in \mathbb{C}$ $\qquad$ $M^3 = \Sigma(K_1, K_2)$

Yuta Nozaki (Meiji Univ.)

joint work with Teruaki Kitano (Soka Univ.)

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§1. Introduction

M : a connected compact n-manifold

$$\begin{array}{ll} R(M) := \mathrm{Hom}(\pi_1(M), SL(2,\mathbb{C})) & \text{the representation variety of } M \\ \quad \big\downarrow \text{proj} & \\ X(M) := R(M) /\!/ SL(2,\mathbb{C}) & \text{the character variety of } M \end{array}$$

Note $\mathrm{proj}(R(M)^{\mathrm{irr}}) \xleftrightarrow{1:1} R(M)^{\mathrm{irr}} / SL(2,\mathbb{C})$

Ex (M = handlebody of genus 2) $\pi_1(M) \cong \langle x, y \mid - \rangle$

$$R(M) \cong SL(2,\mathbb{C})^2 \qquad X(M) \cong \mathbb{C}^3$$

$$\rho \mapsto (\rho(x), \rho(y)) \qquad [\rho] \mapsto (\mathrm{tr}\,\rho(x), \mathrm{tr}\,\rho(y), \mathrm{tr}\,\rho(xy))$$

Thm (Zentner 2018) $M : \mathbb{Z}HS^3 \ (\neq S^3)$, $\chi(M)^{irr} \neq \phi$.

$\downarrow$ using gauge theory

true for $M = \Sigma(K_1, K_2) \implies$ true for $M : \mathbb{Z}HS^3$

"degree 1 map" [Boileau-Rubinstein-Wang 2014]

Def K_1, K_2 : knots in S^3 $\quad E(K_j) := S^3 \setminus \mathrm{Int}\, N(K_j)$

$\Sigma(K_1, K_2) := E(K_1) \cup E(K_2)$ the splice of K_1 and K_2

longitude $\leftrightarrow$ meridian

meridian $\leftrightarrow$ longitude

Note $\Sigma(K_1, K_2)$ is a $\mathbb{Z}HS^3$

Keyword 1 : splice

Thm (Boden - Curtis 2008) $\lambda_{SL(2,\mathbb{C})}(\Sigma(K_1, K_2)) = 0$.

i.e., $X(\Sigma(K_1, K_2))$ has no isolated point.

◎ Today, we consider a function on $X(\Sigma(K_1, K_2))$.

$$\begin{tikzcd} \rho \arrow[r, mapsto] & \text{“}\tau_\rho(M)\text{”} \\ R(M) \arrow[r] \arrow[d] & \mathbb{C} \\ X(M) \arrow[ur, dashed] & \end{tikzcd}$$

Keyword 2: Reidemeister torsion

Convention $\tau_\rho(M) := 0$ if ρ is NOT acyclic

$$\text{acyclic} \overset{\text{def.}}{\Updownarrow} H_*(M; \mathbb{C}^2_\rho) = 0$$

Ex $(M = T^2)$ $X(M) \longrightarrow \mathbb{C}$

$$[\rho] \mapsto \begin{cases} 1 & \text{if } \rho \text{ is acyclic } (\Leftrightarrow \operatorname{tr} \rho(\forall x) \neq 2) \\ 0 & \text{otherwise} \end{cases}$$

Def $RT(M) := \{ \tau_\rho(M) \in \mathbb{C} \mid [\rho] \in X(M) \}$、

Question (Kitano) $\# RT(M) < \infty$ or $= \infty$?

Answer (Kitano)

$\# RT(\text{Seifert mfd}) < \infty$

$\# RT(E(4_1)) = \infty$

$\# RT(E(4_1) \cup_{\text{id}} E(4_1)) = \infty$

$\# RT(\Sigma(4_1, 4_1)) < \infty \xrightarrow{\text{generalize}}$ Main Theorem

Note $\dim_{\mathbb{C}} X(\Sigma(4_1, 4_1)) = 2$ (coming from "bending")

Thm 1 (Kitano-N.) Let K_1, K_2 be knots in S^3 satisfying

① $\gcd(A_{K_1}(L, M), A_{K_2}(M, L)) = 1$,

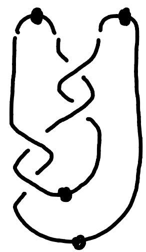
where $A_{K_j}(L, M) \in \mathbb{Z}[L, M]$ is the "A-polynomial" of K_j.

② $\forall C$: an irreducible component of $X(E(K_j))$

- $r_j(C) \neq pt$, where $r_j : X(E(K_j)) \xrightarrow{incl^*} X(\partial E(K_j))$, and
- $\dim_{\mathbb{C}} C \leq 1$.

Then $\#RT(\Sigma(K_1, K_2)) < \infty$.

Cor 1 K_1, K_2 : 2-bridge knots $\Rightarrow \#RT(\Sigma(K_1, K_2)) < \infty$.

$4_1 =$ 

<u>Rem</u> ① is true for $K_1, K_2 \in \{K \mid c(K) \le 10,\ A_K \text{ is known}\}$.

149 knots

② can be replaced with

"$\#\{\tau_\rho(E(K_j)) \mid r_j([\rho]) = [\rho_0]\} < \infty$ for $\forall \rho_0 \in R(T^2)$".

① $\gcd(A_{K_1}(L, M), A_{K_2}(M, L)) = 1$.

② $\forall C$: an irreducible component of $X(E(K_j))$

- $r_j(C) \neq pt$, where $r_j : X(E(K_j)) \xrightarrow{\text{incl}^*} X(\partial E(K_j))$, and
- $\dim_{\mathbb{C}} C \le 1$.

§2. Reidemeister torsion

"Def" Let $\rho \in R(M^n)$: acyclic i.e.,

$$\begin{array}{ccccccc} & & \text{basis} & & & & \\ \cdots \xrightarrow{\partial_{i+1}} & & C_i(M;\mathbb{C}^2_\rho) & \xrightarrow{\partial_i} & C_{i-1}(M;\mathbb{C}^2_\rho) & \rightarrow \cdots & \text{is exact.} \\ & & \cup & & \cup & & \\ & & \operatorname{Im}\partial_{i+1} & & \operatorname{Im}\partial_i & & \\ & & \text{basis} & & \text{basis} & & \end{array}$$

$$\prod_{i=0}^{n} \det\begin{pmatrix}\text{translation} \\ \text{matrix}\end{pmatrix}^{(-1)^i} =: \tau_\rho(M) \in \mathbb{C}^\times$$

the Reidemeister torsion for ρ

Ex $(M = T^2)$ If $\operatorname{tr}\rho({}^\forall x) \neq 2$, then $0 \to C_2 \to C_1 \to C_0 \to 0$ is exact,

$$C_2 = \mathbb{C}^2,\quad C_1 = \mathbb{C}^4,\quad C_0 = \mathbb{C}^2$$

and $\tau_\rho(M) = 1$.

Prop (mulipliсativity) $M = M_1 \cup_{T^2} M_2$, $\rho \in R(M)$.

If ρ and $\rho|_{\pi_1 T^2}$ are acyclic,

then $\rho_j = \rho|_{\pi_1 M_j}$ is acyclic $(j = 1, 2)$

and $\tau_\rho(M) = \tau_{\rho_1}(M_1)\, \tau_{\rho_2}(M_2)$ holds.

In the case $M = \Sigma(K_1, K_2)$,

one can drop the assumption "$\rho|_{\pi_1 T^2}$ is acyclic".

§3. A-polynomial & Proof of Thm 1

$$X(M) = \mathrm{Hom}(\pi_1(M), SL_2)/\!/SL_2$$

$$\begin{array}{ccccc}
 & & & (\mathbb{C}^\times)^2 \subset \mathbb{C}^2 & \\
 & & & \pi \downarrow \quad\quad \cup & \\
 & \text{incl}^* & & \quad\quad \pi^{-1}(\theta\circ r(X(E))) & \overset{\text{closure}}{\rightsquigarrow} \text{curves \& points} \\
 & \| & & & \| \\
X(E) & \xrightarrow{r} X(\partial E) & \underset{1:1}{\xrightarrow{\theta}} & (\mathbb{C}^\times)^2/(L,M)\sim(L^{-1},M^{-1}) & \bigcup_i \{f_i(L,M)=0\} \\
\| & & & & \\
E(K) & [\rho] & \longmapsto & (\rho'(\lambda)_{11}, \rho'(\mu)_{11}) & \\
 & \Psi & & & \\
 & \rho' & & &
\end{array}$$

ρ' : upper triangular

<u>Def</u> (Cooper-Culler-Gillet-Long-Shalen '94)

$$A_K(L,M) := \prod_i f_i(L,M) \in \mathbb{Z}[L,M]$$

the A-polynomial of K

(• gcd(coefficients) = 1
• up to sign)

<u>Ex</u> ($K = 3_1$) $\quad A_K(L,M) = (L-1)(L+M^6)$

Thm 1 (Recall) ① & ② $\Rightarrow$ $\#RT(\Sigma(K_1, K_2)) < \infty$.

$$\text{① \& ②} \overset{\text{Step 2}}{\Longrightarrow} \star \overset{\text{Step 1}}{\Longrightarrow} \#RT(\Sigma(K_1,K_2)) < \infty$$

$\star$ $\left[\begin{array}{l} \exists \text{ a finite subset } X_j \subset X(E(K_j)) \text{ for } j = 1, 2 \\ \text{s.t. } \forall \rho \in X(\Sigma(K_1, K_2)) \quad \rho|_{\pi_1 E(K_j)} \in X_j \end{array}\right]$

Step 1

- ρ: NOT acyclic $\quad \tau_\rho(\Sigma(K_1, K_2)) = 0$.
- ρ: acyclic $\quad$ The multiplicativity & $\star$ imply that

$$\tau_\rho(\Sigma(K_1, K_2)) = \tau_{\rho_1}(E(K_1))\,\tau_{\rho_2}(E(K_2)) \quad \text{for some } \rho_1 \in X_1,\ \rho_2 \in X_2 \quad //$$

① $\gcd(A_{K_1}(L, M),\ A_{K_2}(M, L)) = 1$

② $\forall C \subset X(E(K_j)),\ \ r_j(C) \neq pt\ \ \&\ \ \dim C \leq 1$

☆ $\left[\begin{array}{l} \exists \text{ a finite subset } X_j \subset X(E(K_j)) \\ \text{s.t. } \forall \rho \in X(\Sigma(K_1, K_2))\ \ \rho|_{\pi_1 E(K_j)} \in X_j \end{array}\right]$

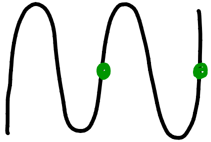
$$\Sigma := \Sigma(K_1, K_2)$$

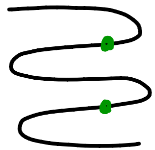
$$\begin{tikzcd} & \Sigma & \\ E(K_1) \arrow[ur] \arrow[rr, phantom, "\circlearrowleft"] & & E(K_2) \arrow[ul] \\ & T^2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\rightsquigarrow X(-)$

$$\begin{tikzcd} & X(\Sigma) \arrow[dl] \arrow[dr] & \\ X(E(K_1)) \arrow[dr, "r_1"'] \arrow[rr, phantom, "\circlearrowleft"] & & X(E(K_2)) \arrow[dl, "r_2"] \\ & X(T^2) & \end{tikzcd}$$

$\mathbb{C}^2$





Step2 (① & ② ⇒ ☆)

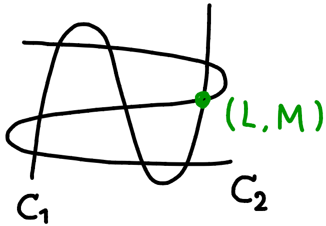
$C_1 := \{(L, M) \mid A_{K_1}(L, M) = 0\}$

$C_2 := \{(L, M) \mid A_{K_2}(M, L) = 0\}$

Let $X_j := r_j^{-1}(C_1 \cap C_2)$ for $j = 1, 2$

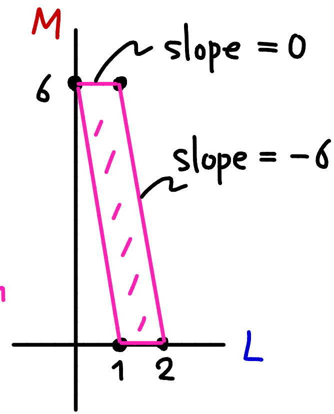
Then, for $\forall \rho \in X(\Sigma)$, $\rho|_{\pi_1 E(K_j)} \in X_j$ (by ②)

Here, $\left.\begin{array}{l} \text{by ①, } \#(C_1 \cap C_2) < \infty \\ \text{by ②, } \# r_j^{-1}((L, M)) < \infty \end{array}\right\} \rightsquigarrow \# X_j < \infty$ ▨



§4. Slopes & Proof of Cor 1

Ex ($K = 3_1 = T_{-2,3}$) $A_K(L,M) = (L-1)(L+M^6)$



① Slopes of the sides

$L^2 + LM^6 - L - M^6 \rightsquigarrow$ lattice points $\underset{\text{convex hull}}{\rightsquigarrow}$ Newton polygon

$SS(N(A_K)) = \{-6, 0\} \subset \mathbb{Q} \cup \{\infty\}$.

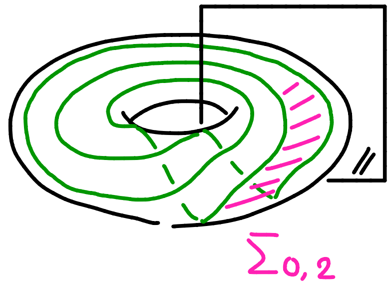
② Boundary slopes (e.g., Seifert surf.)

$\Sigma_{g,b}$: a properly emb. incompressible surf.

$\partial\Sigma_{0,2} = \gamma_1 \sqcup \gamma_2$ $\pm[\gamma_i] = \lambda - 6\mu \in H_1(\partial E(K))$

slope $= \frac{-6}{1}$

$BS(K) = \{-6, 0\}$.



Thm (CCGLS '94) $SS(N(A_K)) \subset BS(K)$ for $\forall K$.

Thm (Hatcher-Thurston '85)

K: a 2-bridge knot $\Rightarrow$ $BS(K) \subset 2\mathbb{Z}$.

By the above theorems,

Lem 1 K_1, K_2: 2-bridge knots

$\Rightarrow SS(N(A_{K_1})) \cap SS(N(A_{K_2}))^{-1} = \phi$.

(For a subset $S \subset \mathbb{Q} \cup \{\infty\}$, $S^{-1} := \{ s^{-1} \mid s \in S \}$)

Cor 1 (Recall) K_1, K_2: 2-bridge knots $\Rightarrow \# RT(\Sigma(K_1, K_2)) < \infty$

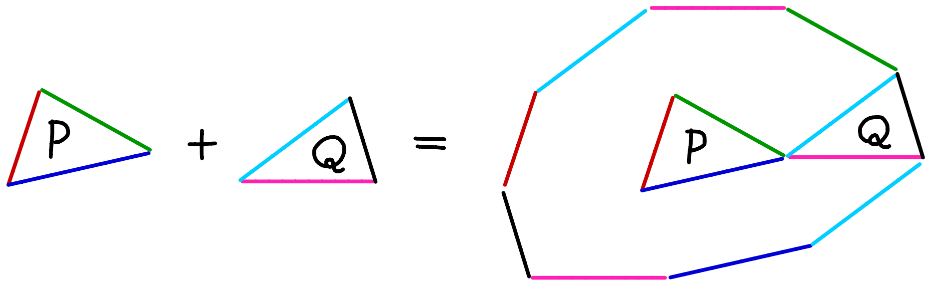
To prove Cor 1, we need $\begin{cases} \text{Lem1 in the previous slide \&} \\ \text{Lem2 below.} \end{cases}$

<u>Fact</u> P, Q : convex polygons.

Then, $SS(P+Q) = SS(P) \cup SS(Q)$.

Minkowski sum

<u>Ex</u>



Fact shows the following lemma:

<u>Lem 2</u> $f_1, f_2 \in \mathbb{Z}[L, M]$.

$SS(N(f_1)) \cap SS(N(f_2))^{-1} = \phi \implies \gcd(f_1(L,M), f_2(M,L))$ is a monomial.

<u>Cor 1</u> (Recall) K_1, K_2 : 2-bridge knots $\Rightarrow \# RT(\Sigma(K_1, K_2)) < \infty$

Proof. It suffices to check ① & ② in Thm 1.

① $\gcd(A_{K_1}(L, M), A_{K_2}(M, L)) = 1$.

∵ By Lem 1, $SS(N(A_{K_1})) \cap SS(N(A_{K_2}))^{-1} = \phi$

Hence, by Lem 2, $\gcd(A_{K_1}(L, M), A_{K_2}(M, L))$ is a monomial.

Since $L, M \nmid A_K(L, M)$ in general, $\gcd = 1$ //

② $\forall C \subset X(E(K_j))$, $r(C) \neq pt$ & $\dim C \leq 1$.

∵ Some results on "Riley polynomial" of K_j // ▨

Future research

- Study / Drop the assumptions ①, ② in Thm 1.

- (untwisted) splice, $\gcd(A_{K_1}(L,M),\ A_{K_2}(M,L)) = 1$?

  $\downarrow$

  twisted splice, $\gcd($ 〃 $,\ A_{K_2}(L^p M^q,\ L^r M^s)) = 1$?

Ex $M = E(4_1) \cup_{\mathrm{id}} E(4_1)$ $(p=1,\ q=r=0,\ s=1)$

Then, $\begin{cases} \gcd(A_{4_1}(L,M),\ A_{4_1}(L,M)) = A_{4_1}(L,M) \neq 1 \\ \#RT(M) = \infty \end{cases}$