Complex analysis with Thurston theory in the Teichmüller theory

Hideki Miyachi

Kanazawa University

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at しいのき迎賓館（金沢）

Welcome to Kanazawa !

Sightseeing

- Kenroku-en Garden : One of the three noted gardens in Japan.


- Kanazawa Castle Park:


Welcome to Kanazawa !

- Higashi Chaya District :


Foods

- Oden (Stew), Sushi (Seafoods), etc.

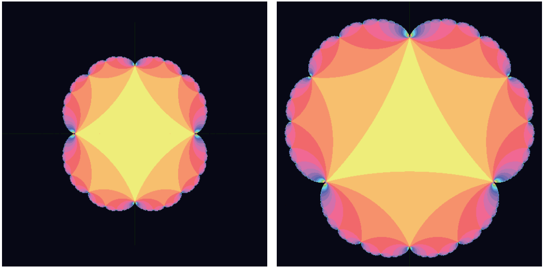

Enjoy at Kanazawa !!

Section 1

Motivation

Question 1 (Long-standing problem (it used to be popular))

Study the shape of the deformation spaces of Kleinian groups.



Left : Bers slice for a square torus

Right : Bers slice for a hexagonal torus

(Courtesy of Professor Yasushi Yamashita)

Many pictures of the deformation spaces are drawn. All pictures are very impressed and yield many questions. For instance,

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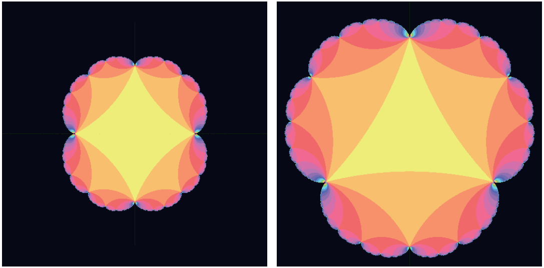

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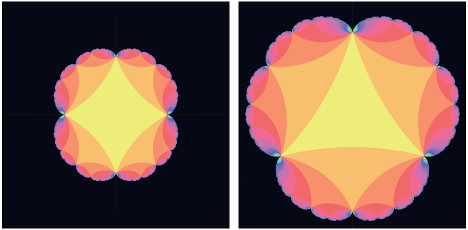
Problem 1 (McMullen, Annals of Mathematics Studies 142 (1996))

Is the boundary of a Bers slice self-similar about the fixed points of pseudo-Anosov actions?

- The boundary of the deformation space separates the representations into "discrete representations" and "non-discrete representations".
- Thurston's program (in '78) clarifies what the separation is : The Ending Lamination Theorem (settled by Brock, Canary and Minsky) tells us that the boundary (the separation) is parametrized by the end-invariants (Topological data $+\alpha$).
- The study of the shape will clarify

  How discrete and non-discrete representations are separated.

- Problems on the shape are problems after Thurston's program.



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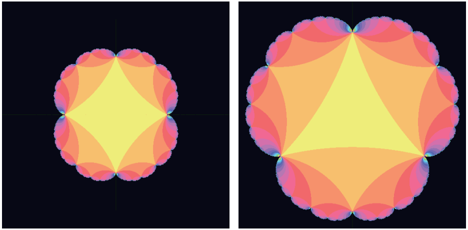
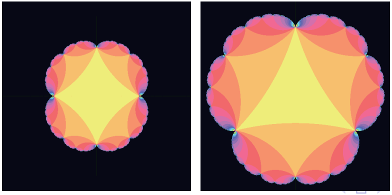


Our strategy

- Trace functions are holomorphic functions on the ambient space and are local charts around the boundary.
- Holomorphic functions are very smooth (infinitesimally, complex linear). The local behavior of holomorphic functions on the bdy may reflect the (local) "shape" of the bdy (e.g. self-similality).
- For understanding the relation between the "shape" and end-invariants (Topology$+\alpha$), we pose

Question 2 (My long-standing problem (of course, not popular))

Study holomorphic functions on the Bers closure from Thurston theory.



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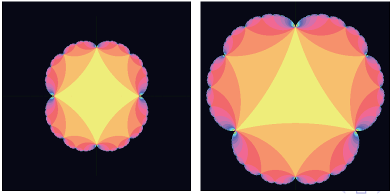


Section 2

Notation

Notation

From Function theory (and Theory of Kleinian groups):

- Σ_g : a closed orientable surface of genus g (≥ 2).
- $\mathcal{T}_g$: the Teichmüller space of genus g.
- d_T : the Teichmüller distance.
- $\mathcal{T}^B_{x_0}$: the Bers slice with center $x_0 \in \mathcal{T}_g$ $(\subset A^2(\mathbb{H}^*, \Gamma_0) \cong \mathbb{C}^{3g-3})$.
- $\partial\mathcal{T}^B_{x_0}$: the Bers boundary.

From Thurston theory:

- $\mathcal{ML}$, $\mathcal{PML}$: measured laminations and projective measured laminations on Σ_g.
- $\mathrm{Ext}_x(F)$: the extremal length of $F \in \mathcal{ML}$.
- $\mathcal{SML}_x$: the unit sphere $\{F \in \mathcal{ML} \mid \mathrm{Ext}_x(F) = 1\}$ $(x \in \mathcal{T}_g)$. $\mathcal{SML}_x \cong \mathcal{PML}$ via the projection $\mathcal{ML} - \{0\} \to \mathcal{PML}$.

Notation

From Thurston theory (continued):

- $\mathcal{SML}_x^{mf}$: a subset of $\mathcal{SML}_x$ consisting of minimal, filling measured laminations.
- $\mathcal{SML}_x^{ue}$: a subset of $\mathcal{SML}_x^{mf}$ consisting of uniquely ergodic measured laminations.
- $\mathcal{PML}^{mf}$, $\mathcal{PML}^{ue}$: corresponding subsets to $\mathcal{SML}_x^{mf}$ and $\mathcal{SML}_x^{ue}$.

Proposition 1 (follows from two big theorems: DLT and ELT)

For $x \in \mathcal{T}_g$, there is a continuous map

$$\Xi_x \colon \mathcal{SML}_x^{mf} \cong \mathcal{PML}^{mf} \xrightarrow{onto} \partial^{mf}\mathcal{T}_{x_0}^B \subset \partial\mathcal{T}_{x_0}^B$$

which induces a homeomorphism $\mathcal{SML}_x^{ue} \cong \mathcal{PML}^{ue} \to \partial^{ue}\mathcal{T}_{x_0}^B$.

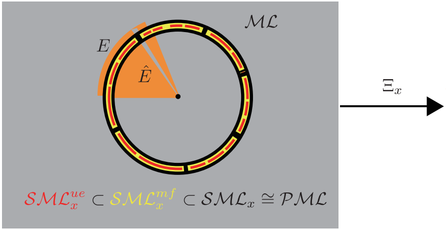
Notation

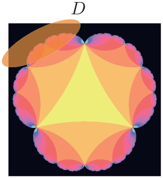
From Thurston theory (continued):

- μ_{Th} : The Thurston measure (MCG-invariant measure on $\mathcal{ML}$).
- $\hat{\mu}^x_{Th}$: A Borel measure on $\mathcal{SML}_x$ defined by

$$\hat{\mu}^x_{Th}(E) = \frac{\mu_{Th}(\{tF \mid F \in E, 0 \leq t \leq 1\})}{\mu_{Th}(\{tF \mid F \in \mathcal{SML}_x, 0 \leq t \leq 1\})} \quad (E \subset \mathcal{SML}_x).$$

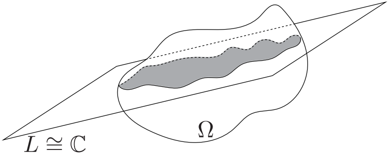
- $\boldsymbol{\mu}^B_x$: the pushforward measure defined from $\hat{\mu}^x_{Th}$ via Ξ_x $(x \in \mathcal{T}_g)$, which is a Borel measure supported on $\partial\mathcal{T}^B_{x_0}$.





Complex analysis and Demailly's theory

- An upper-semicontinuous function u on a domain $\Omega \subset \mathbb{C}^N$ is said to be plurisubharmonic if for any complex line $L(\cong \mathbb{C})$ in $\mathbb{C}^N$ with $\Omega \cap L \neq \emptyset$, $u\,|_{\Omega\cap L}$ is subharmonic on $\Omega \cap L$.
- A function u on a domain $\Omega \subset \mathbb{C}^N$ is pluriharmonic if u and $-u$ are plurisubharmonic. Any pluriharmonic function is locally the real part of a holomorphic function.
- A bounded domain Ω in $\mathbb{C}^N$ is said to be hyperconvex if it admits a continuous and non-positive plurisubharmonic exhaustion u (i.e. $\{x \in \Omega \mid u(x) < r\}$ is relatively compact for each $r < 0$).



Complex analysis and Demailly's theory

Theorem 2 (Demailly (1987))

Let Ω be a hyperconvex domain in $\mathbb{C}^N$. There is a positive Borel function $\mathbb{P}_\Omega$ on $\Omega \times \Omega \times \partial\Omega$ and a family $\{\mu_x^\Omega\}_{x\in\Omega}$ of Borel measures supported on $\partial\Omega$ such that

- $d\mu_y^\Omega = \mathbb{P}_\Omega(x, y, \cdot)d\mu_x^\Omega$ *for* $x, y \in \Omega$; *and*
- *any pluriharmonic function u on Ω which is continuous on $\overline{\Omega}$ satisfies*

$$u(x) = \int_{\partial\Omega} u(z)d\mu_x^\Omega(z) = \int_{\partial\Omega} u(z)\mathbb{P}_\Omega(x_0, x, z)d\mu_{x_0}^\Omega(z)$$

for fixed $x_0 \in \Omega$. Thus, $\mathbb{P}_\Omega$ is the Poisson kernel for pluriharmonic functions and μ_x^Ω is the pluriharmonic measure at $x \in \Omega$.

Complex analysis and Demailly's theory

Theorem 3 (Demailly (1987))

Any hyperconvex domain $\Omega \subset \mathbb{C}^N$ admits a unique pluricomplex Green function g_Ω.

We do not give the definition of the pluricomplex Green function here, but the pluricomplex Green function is very important to calculate the Poisson kernel. Roughly speaking,

$$\mathbb{P}_\Omega(x, y, \zeta) = \lim_{z \to \zeta} \left(\frac{g_\Omega(y, z)}{g_\Omega(x, z)} \right)^N \tag{1}$$

for $\zeta \in \partial\Omega$ (if the limit exists).

Complex analytic property on Teichmüller space

- Teichmüller space is hyperconvex (Krushkal). For instance,

$$u(x) = -\frac{1}{\mathrm{Ext}_x(F) + \mathrm{Ext}_x(G)}$$

is continuous non-negative plurisubharmonic exhaustion on $\mathcal{T}_{x_0}^B \cong \mathcal{T}_{g,m}$, when $F, G \in \mathcal{ML}$ fill Σ_g up (M).

- The pluricomplex Green function is obtained as

$$g_{\mathcal{T}_{g,m}}(x,y) = \log\tanh d_T(x,y)$$

for $x, y \in \mathcal{T}_{g,m}$ (Krushkal, M).

- The Bers slice $\mathcal{T}_{x_0}^B$ is polynomially convex (Shiga): holomorphic functions on the ambient space is dense in the space of holomorphic functions on $\mathcal{T}_{x_0}^B \cong \mathcal{T}_{g,m}$ in the compact open topology.

Section 3

Main results

Poisson integral formula

Theorem 1 (Pluriharmonic measure and Poisson kernel)

For the Bers slice $\mathcal{T}^B_{x_0}$, we have

$$\boldsymbol{\mu}^B_y(E) = \int_E \mathbb{P}(x,y,\varphi)\,d\boldsymbol{\mu}^B_x(\varphi) \quad (E \subset \partial\mathcal{T}^B_{x_0})$$

where $\mathbb{P}$ is a measurable function on $\mathcal{T}_g \times \mathcal{T}_g \times \partial\mathcal{T}^B_{x_0}$ defined by

$$\mathbb{P}(x,y,\varphi) = \begin{cases} \left(\dfrac{\mathrm{Ext}_x(F_\varphi)}{\mathrm{Ext}_y(F_\varphi)}\right)^{3g-3} & (\varphi \in \partial^{ue}\mathcal{T}^B_{x_0}), \\ 1 & (\textit{otherwise}) \end{cases} \qquad (2)$$

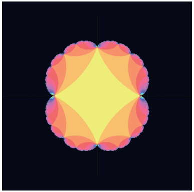
where F_φ is the measured lamination whose support is the ending lamination of the Kleinian manifold associated to φ.

Poisson integral formula

Theorem 2 (Poisson integral formula)

Let u be a pluriharmonic function on $\mathcal{T}_g$ which is continuous on the Bers closure. Then,

$$u(x) = \int_{\partial\mathcal{T}_{x_0}^B} u(\varphi) d\boldsymbol{\mu}_x^B(\varphi) = \int_{\partial\mathcal{T}_{x_0}^B} u(\varphi)\mathbb{P}(x_0, x, \varphi) d\boldsymbol{\mu}_{x_0}^B(\varphi)$$



Polynomially convexity implies :

Any holomorphic function on $\mathcal{T}_{x_0}^B \cong \mathcal{T}_{g,m}$ is approximated by holomorphic functions represented by the Poisson integral formula.

Poisson integral formula

Theorem 3 (Schwarz type theorem)

Let V be a $\boldsymbol{\mu}_{x_0}^B$-integrable function on $\partial\mathcal{T}_{x_0}^B$, which is continuous on $\partial^{ue}\mathcal{T}_{x_0}^B$. Suppose that

$$\int_{\partial\mathcal{T}_{x_0}^B} V(\varphi)\overline{\partial}\mathbb{P}(x_0, x, \varphi)d\boldsymbol{\mu}_{x_0}^B(\varphi) = 0$$

as $(0,1)$-form on $\mathcal{T}_g$. Then

$$P[V](x) = \int_{\partial\mathcal{T}_{x_0}^B} V(\varphi)\mathbb{P}(x_0, x, \varphi)d\boldsymbol{\mu}_{x_0}^B(\varphi).$$

is a holomorphic function on $\mathcal{T}_g$ satisfying

$$\lim_{x\to\varphi} P[V](x) = V(\varphi) \quad (\varphi \in \partial^{ue}\mathcal{T}_{x_0}^B).$$

Why Poisson integral formula?

- Poisson integral formula and Schwarz type theorem gives an interaction

  $$\{\text{Some hol.fns on } \mathcal{T}_g\} \rightleftharpoons \{\text{Meas. fns on } \partial\mathcal{T}^B_{x_0} \text{ with some condition}\}$$

  where the condition of a function V on $\partial\mathcal{T}^B_{x_0}$ is

  $$\int_{\partial\mathcal{T}^B_{x_0}} V(\varphi)\overline{\partial}\mathbb{P}(x_0, \cdot, \varphi)d\boldsymbol{\mu}^B_{x_0}(\varphi) = 0$$

  as a $(0,1)$-form on $\mathcal{T}_g$.
- I suspect that the interaction induces the isomorphism

  $$\{\text{Bdd hol.fns on } \mathcal{T}_g\} \rightleftharpoons \{L^\infty\text{-fns on } \partial\mathcal{T}^B_{x_0} \text{ with the condition}\}$$

  as complex Banach spaces (my ongoing project).
- Notice that every holomorphic function on the Bers closure is bounded (because the Bers closure is compact).

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Why are boundary functions important?

- As noticed before, hol.fns are very smooth (infinitesimally, complex linear) and the local behavior of hol.fns on the bdy reflects the (local) "shape" of the bdy (e.g. self-similality).
- The Poisson integral formula is rewritten as

$$P[V](x) = \int_{\mathcal{PMF}} \hat{V}([H]) \left(\frac{\mathrm{Ext}_{x_0}(H)}{\mathrm{Ext}_x(H)} \right)^{3g-3} d\hat{\mu}_{Th}^{x_0}([H])$$

where $\hat{V}$ is a measurable fn on $\mathcal{SML}_{x_0} \cong \mathcal{PML}$ (cont.on $\mathcal{SML}_{x_0}^{mf} \cong \mathcal{PML}^{mf}$ if V is cont.on the Bers closure) defined by

$$\hat{V} = V \circ \Xi_{x_0}.$$

Connection between Complex analysis and Thurston theory

Bdy fns are fns on $\mathcal{PML}$ (where is from the topological aspect!).

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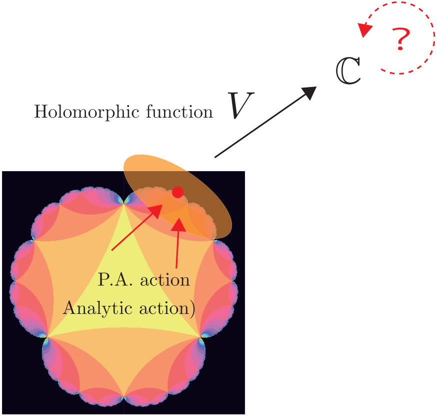

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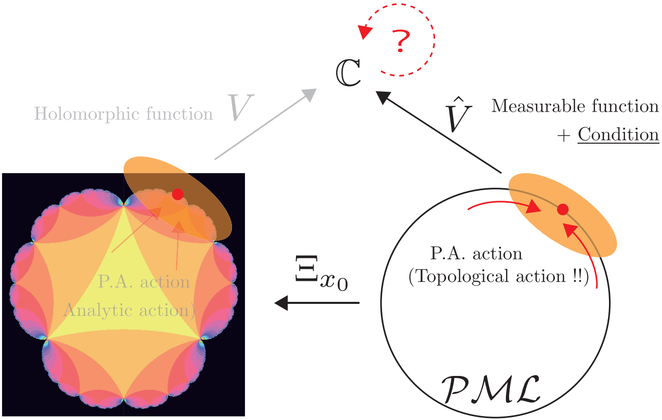

Problem : Is the condition easy to use?

- The condition for boundary functions is

$$\int_{\partial\mathcal{T}^B_{x_0}} V(\varphi)\overline{\partial}\mathbb{P}(x_0,\cdot,\varphi)d\boldsymbol{\mu}^B_{x_0}(\varphi)=0.$$

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$$\int_{\mathcal{PMF}}\hat{V}([H])\left(\frac{\mathrm{Ext}_{x_0}(H)}{\mathrm{Ext}_x(H)}\right)^{3g-3}\frac{\overline{q_{H,x}}}{\|q_{H,x}\|}d\hat{\mu}^{x_0}_{Th}([H])=0 \quad (\forall x\in\mathcal{T}_g)$$

where

$$\hat{V}=V\circ\Xi_{x_0} \quad (\text{defined on } \mathcal{SML}_{x_0}\cong\mathcal{PML})$$

by Gardiner's formula and the definition of the measure $\boldsymbol{\mu}^B_{x_0}$, where $q_{H,x}$ is the Hubbard-Masur differential on $x\in\mathcal{T}_g$ for $H\in\mathcal{ML}$.

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How do we rewrite the condition for use?

- To analyse the condition of bdy fns:

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  we (probably) need to study the "infinitesimal structure" of the Bers boundary for localizing the condition.
- For this, we will (or may) need to study (I suspect)
  - the "infinitesimal" behavior of the map $\Xi_{x_0} : \mathcal{PML}^{mf} \to \partial^{mf}\mathcal{T}^B_{x_0}$ ($\mathcal{PML}$ has PL structure)
  - the "infinitesimal structure" of the ending lamination space (the space of end-invariants).
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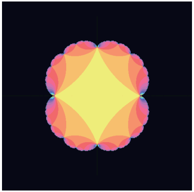
Summary

Question 4 (Main problem (already appeared))

Study the shape of the deformation space of Kleinian groups

- Holomorphic functions on the ambient space are local charts on the boundary.
- Holomorphic functions on the Bers closures are represented by the Poisson integral.
- Boundary functions are thought of as measurable functions on $\mathcal{PML}$.
- The "local complexity" of the Bers boundary reflects the local behavior of the boundary functions.
- We are looking for the condition of boundary functions for discussing the local behavior.



Idea of the proof

Idea

Use Demailly's theory.

- Determine the Poisson kernel
  - Applying Extremal length geometry (Kerckhoff, Gardiner-Masur, M). We can see

$$\lim_{z\to\varphi}\frac{g_{\mathcal{T}_{g,m}}(y,z)}{g_{\mathcal{T}_{g,m}}(x,z)}=\lim_{z\to\varphi}\frac{\log\tanh d_T(y,z)}{\log\tanh d_T(x,z)}=\frac{\mathrm{Ext}_x(F_\varphi)}{\mathrm{Ext}_y(F_\varphi)}$$

for $\varphi\in\partial^{ue}\mathcal{T}^B_{x_0}$ (cf. (1)). Notice the Kerckhoff formula

$$d_T(x,y)=\log\sup_{F\in\mathcal{MF}-\{0\}}\frac{\mathrm{Ext}_x(F)}{\mathrm{Ext}_y(F)}\quad(x,y\in\mathcal{T}_{g,m}).$$

- Pluriharmonic measure coincides with the pushforward measure.
  - Pluriharmonic measure is absolutely continuous w.r.t. the pushforward measure (+ Ergodicity of the MCG-action on $\mathcal{PML}$ (Masur)).
  - (Technical part) Comparing the pluriharmonic measure and measures from extremal length.

Section 4

Application

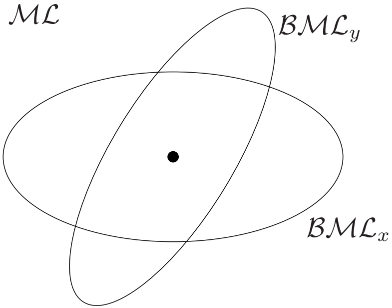
Hubbard-Masur function is constant

Corollary 4 (Hubbard-Masur function is constant (Mirzakhani-Dumas))

The Hubbard-Masur function

$$\mathcal{T}_{g,m} \ni x \mapsto \mathrm{Vol}_{Th}(x) = \mu_{Th}(\mathcal{BML}_x)$$

is a constant function where $\mathcal{BML}_x = \{F \in \mathcal{ML} \mid \mathrm{Ext}_x(F) \leq 1\}$.



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is a constant function where $\mathcal{BML}_x = \{F \in \mathcal{ML} \mid \mathrm{Ext}_x(F) \leq 1\}$.

Proof.

$$\begin{aligned} 1 &= \int_{\mathcal{PML}} \left(\frac{\mathrm{Ext}_{x_0}(F)}{\mathrm{Ext}_x(F)}\right)^{3g-3} d\hat{\mu}^{x_0}_{Th}([F]) \\ &= \frac{1}{\mathrm{Vol}_{Th}(x_0)} \int_{\mathcal{BML}_{x_0}} \left(\frac{\mathrm{Ext}_{x_0}(F)}{\mathrm{Ext}_x(F)}\right)^{3g-3} d\mu_{Th}(F) = \frac{\mathrm{Vol}_{Th}(x)}{\mathrm{Vol}_{Th}(x_0)}. \end{aligned}$$

□

Representation of holomorphic quadratic differentials

Let V be a pluriharmonic function on $\mathcal{T}_{g,m}$ which is continuous on the Bers closure. Then,

$$V(x) = \int_{\mathcal{PML}} \hat{V}([F]) \left(\frac{\mathrm{Ext}_{x_0}(F)}{\mathrm{Ext}_x(F)} \right)^{3g-3} d\hat{\mu}_{Th}^{x_0}([F]),$$

$$(\partial V)_x = (3g-3) \int_{\mathcal{PML}} \hat{V}([F]) \left(\frac{\mathrm{Ext}_{x_0}(F)}{\mathrm{Ext}_x(F)} \right)^{3g-3} \frac{q_{F,x}}{\|q_{F,x}\|} d\hat{\mu}_{Th}^{x_0}([F]),$$

because of the Gardiner formula:

$$(\partial \mathrm{Ext}_{\cdot}(F))_x = -q_{F,x} \quad (x \in \mathcal{T}_{g,m})$$

and $\|q_{F,x}\| = \mathrm{Ext}_x(F)$, where $q_{F,x}$ is the Hubbard-Masur differential for $F \in \mathcal{ML}$ at $x \in \mathcal{T}_{g,m}$.

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Representation of holomorphic quadratic differentials

For $\gamma \in \pi_1(\Sigma_g)$, let $\Theta_{\gamma,x}$ be a holomorphic quadratic differential with

$$d\,\mathrm{hLeng}_\gamma[v] = \mathrm{Re}\int_M \mu\Theta_{\gamma,x}$$

for $v = [\mu] \in T_x\mathcal{T}_g$ at $x = (M, f) \in \mathcal{T}_{g,m}$ (Gardiner, Wolpert).

Theorem 5

$$\Theta_{\gamma,x} = \frac{3g-3}{\sinh(\mathrm{hLeng}_\gamma(x)/2)}\int_{\mathcal{PML}^{mf}} \mathrm{tr}^2(\rho_{\varphi_{F,x}}(\gamma))\frac{q_{F,x}}{\|q_{F,x}\|}d\hat{\mu}^{x_0}_{Th}([F])$$

where $\varphi_{F,x} \in \partial\mathcal{T}^B_{x_0}$ ($[F] \in \mathcal{PMF}^{mf}$) is the totally degenerate group whose ending lamination is the support of F.

Thank you very much for your attention (^ ^)/ !!

ご静聴ありがとうございました.