

Angles, 1-coboundaries and identities. Kanazawa 2019

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Foreword

This talk is about identities for functions on Teichmeuller space and more generally relations between classes of functions.

We will discuss three ways of proving these relations

1. Geometric
2. (Discrete) Line integrals in a graph
3. Differentiation on Teichmeuller space.

Part I

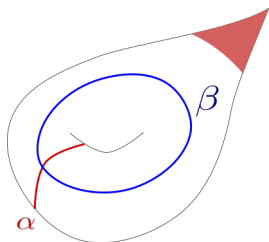
Introduction

One holed torus, notation

- ▶ Σ is a hyperbolic one holed torus, totally geodesic boundary $\partial\Sigma$ or a puncture
- ▶ $\mathcal{T}(\Sigma)$ its Teichmueller space
- ▶ α, β a pair of closed simple geodesics that meet in a single point x .
- ▶ $\alpha, \beta \in \pi_1(\Sigma)$ are a pair of generators
- ▶ l_α the length of α and τ_α the Fenchel-Nielsen parameter for α
- ▶ $\partial/\partial\tau_\alpha$ the Fenchel-Nielsen parameter for α

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Closed Geodesics

Lemma (Pinching α)

Let Σ be a hyperbolic surface and $\alpha, \beta \subset \Sigma$ a pair of distinct closed (simple) geodesics which intersect in x with θ the angle between them at x . Then

$$\sinh(l_\alpha/2) \sinh(l_\beta/2) \geq 1.$$

So that if $l_\alpha \rightarrow 0$ then $l_\beta \rightarrow \infty$.

First relation: Let α, β a pair of closed simple geodesics that meet in a single point on a torus with totally geodesic boundary γ . Then

$$\sinh(l_\alpha/2) \sinh(l_\beta/2) \sin \theta = \cosh(l_\gamma/4).$$

Part II

Identities

Identity for embedded pants

Σ has a single boundary component of length $\ell(\delta) \geq 0$

- ▶ Punctured torus $\ell(\delta) = 0$

$$\sum_{\alpha} \frac{1}{1 + e^{\ell(\alpha)}} = \frac{1}{2}$$

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- ▶ One-holed torus

$$\sum_{\alpha} 2 \log \left(\frac{1 + e^{\frac{1}{2}(-\ell(\alpha) + \ell(\delta))}}{1 + e^{\frac{1}{2}(-\ell(\alpha) - \ell(\delta))}} \right) = \ell(\delta)$$

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- ▶ One-holed genus g

$$\sum_P 2 \log \left(\frac{1 + e^{\frac{1}{2}(-\ell(\alpha) - \ell(\beta) + \ell(\delta))}}{1 + e^{\frac{1}{2}(-\ell(\alpha) - \ell(\beta) - \ell(\delta))}} \right) = \ell(\delta)$$

P is an embedded pair of pants with waist δ and legs α, β

Properties of Gap functions

$$\mathcal{D}(x, y, z) = 2 \log \left(\frac{1 + e^{\frac{1}{2}(x-y-z)}}{1 + e^{\frac{1}{2}(-x-y-z)}} \right)$$

- ▶ \mathcal{D} is monotone increasing in $x = \ell_\delta$ decreasing in y, z
 \Rightarrow if $\partial\Sigma$ gets longer then so do closed simple geodesics.
- ▶ Under "rescaling" \mathcal{D} converges to a piecewise linear function.

$$\frac{2}{t} \log \left(\frac{1 + e^{\frac{1}{2}(x-y-z)t}}{1 + e^{\frac{1}{2}(-x-y-z)t}} \right) \rightarrow \begin{cases} x - y - z & y + z < x \\ 0 & y + z \geq x \end{cases}$$

Rescaling can be used to "tropicalise" the surface as boundary length $\rightarrow \infty$.

Motivation

- ▶ Work of Hu, Tan, Zhang : A NEW IDENTITY FOR $SL(2, \mathbb{C})$ -CHARACTERS OF THE ONCE PUNCTURED TORUS GROUP
Dedicated to Professor Sadayoshi Kojima on the occasion of his sixtieth birthday.
- ▶ A curious paper of Zagier (in French) about relations between special values of the zeta function.

Hu, Tan, Zhang

Theorem 1.1. *Let $\rho : \pi_1(T) \rightarrow \mathrm{SL}(2, \mathbf{C})$ be an irreducible representation and let $\mu = \tau + 2$ where $\tau \in \mathbf{C}$ is the peripheral trace of ρ . If ρ satisfies the Bowditch conditions then*

$$\sum_{X \in \Omega} h_\mu(x) = \frac{1}{2}, \quad (2)$$

where the infinite sum converges absolutely, $x = \mathrm{tr} \rho(X)$, and the function $h_\mu : \mathbf{C} \setminus \{[-2, 2], \pm\sqrt{\mu}\} \rightarrow \mathbf{C}$ is defined by

$$h_\mu(x) = \frac{1}{2} \left(1 - \frac{x^2 - \frac{2}{3}\mu}{x^2 - \mu} \sqrt{1 - \frac{4}{x^2}} \right). \quad (3)$$

In terms of lengths they actually show that the following sum equals 1:

$$\sum_{\alpha} \left(1 - \frac{\sinh(\frac{\ell_{\alpha}}{2}) \cosh(\frac{\ell_{\alpha}}{2})}{\cosh^2(\frac{\ell_{\alpha}}{2}) - \cosh^2(\frac{\ell_{\gamma}}{4})} \right) + \sum_{\beta} \frac{\tanh(\frac{\ell_{\beta}}{2})}{\cosh^2(\frac{\ell_{\beta}}{2}) - \cosh^2(\frac{\ell_{\gamma}}{4})}.$$

The first sum is over **all** simple closed geodesics and the second only those passing through the same pair of Weierstrass points.

QUELQUES CONSÉQUENCES SURPRENANTES DE LA COHOMOLOGIE DE $SL_2(\mathbb{Z})$

DON ZAGIER

Je vais parler de beaucoup de choses différentes. J'avais l'intention de faire un petit plan, une sorte de "Leitfaden", et j'ai commencé, mais il est devenu bidimensionnel, puis tridimensionnel et puis quadridimensionnel, et j'ai abandonné. Il faudra que vous vous rendiez compte vous-mêmes de quoi je parle au cours de mon exposé. Il est assez peu probable que vous y réussissiez complètement, car il va apparaître des thèmes assez variés et les liens entre eux ne seront sans doute pas toujours très clairs. Mais il y a une chose qui va apparaître tout le temps et que vous allez reconnaître à chaque reprise : des formules dans lesquelles il y a un X quelque part, un $X + 1$, et un $1 + 1/X$, liés par une relation du type

$$F(X) \approx F(1 + X) + F(1 + 1/X) \quad (*)$$

Part III

Geometric construction

Geometric Decompositions

Decomposition:

some space $X = (\sqcup\{\text{geometric pieces}\}) \sqcup \{\text{negligible}\}$

- ▶ $X = \partial\Sigma$
- ▶ $X = \text{unit tangent bundle } \Sigma,$
negligible = geodesics that stay in convex core.

Pants

$$\sum_{\alpha} 2 \log \left(\frac{1 + e^{\frac{1}{2}(\ell(\alpha) + \ell(\beta) - \ell(\delta))}}{1 + e^{\frac{1}{2}(\ell(\alpha) + \ell(\beta) + \ell(\delta))}} \right) = \ell(\delta)$$

What is the associated decomposition of the surface ?

Gap decomposition of δ

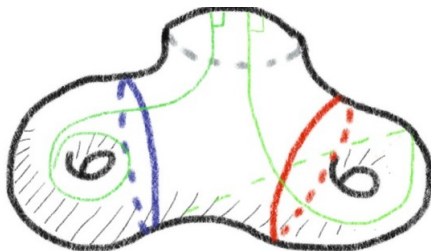
Define $X \subset \delta$ to be the set of x starting points for $\gamma_x :=$ geodesic leaving δ at right angles which

- ▶ is simple
- ▶ stays in the surface forever.

Gap decomposition of δ

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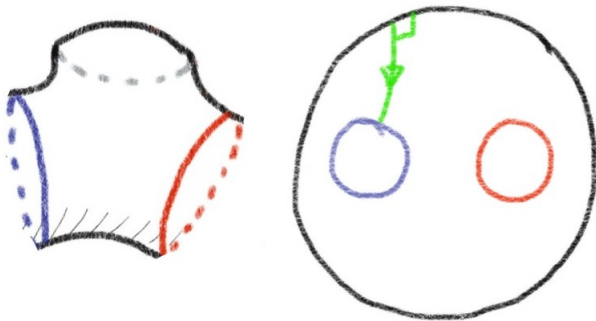


Most geodesics don't stay inside forever.

Gap decomposition of the boundary geodesic δ

The geodesic ray γ_x

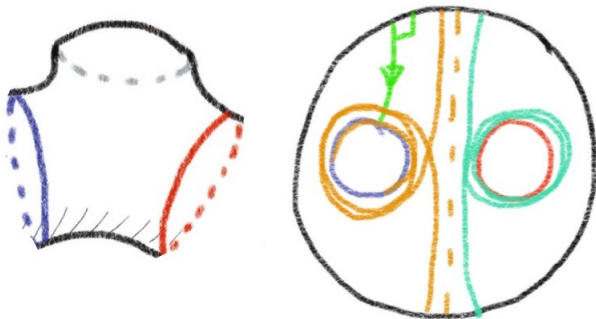
- ▶ either exits a pair of pants by one of the boundaries α, β .
- ▶ or spirals to one of the boundaries α, β .



Lemma

There are a pair of intervals $\subset \delta$ which contain no point of X

Proof : the \mathcal{D} gaps



Part IV

Renormalised line integral

Markoff Tree

Solutions in positive integers of

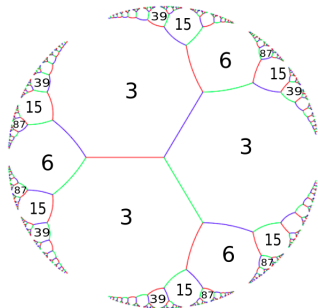
$$X^2 + Y^2 + Z^2 - XYZ = 0$$

obviously symmetric i.e. invariant under cyclic permutation

$$U : (X, Y, Z) \mapsto (Y, Z, X).$$

Theorem (Cohn)

Triples of solutions are 1-1 with triples of (cosh length/2 of) closed geodesics in Σ that meet pairwise in exactly one point.



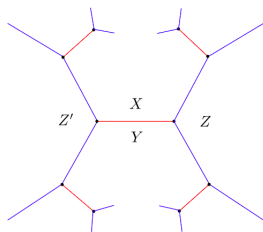
Markoff Tree

If $(X, Y, Z = Z^\pm)$ is a pair of solutions of

$$0 = X^2 + Y^2 + Z^2 - XYZ = Z^2 - XY(Z) + (X^2 + Y^2)$$

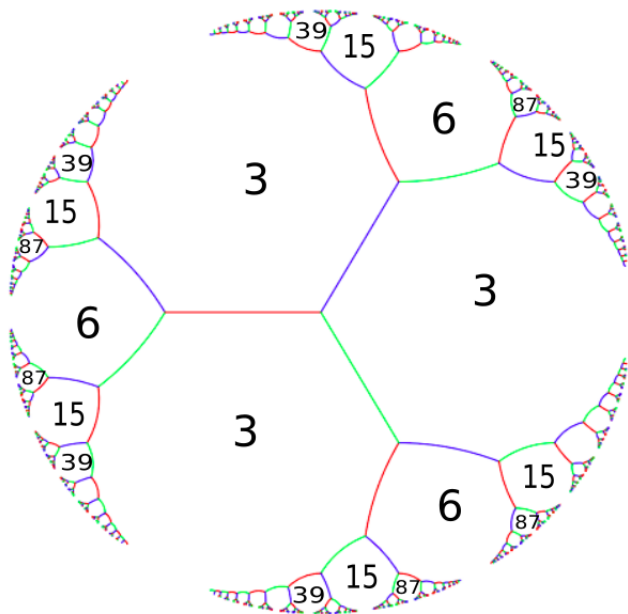
then $Z^+ + Z^- = XY$ and there is a root swapping involution.

$$S : (X, Y, Z^+) \mapsto (X, Y, Z^-)$$



$$\mathcal{MCG}(\Sigma) \simeq \mathrm{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \simeq \langle S \rangle * \langle U \rangle$$

Labelled complementary regions



Coboundaries

If

- ▶ X is a space
- ▶ $T : X \mapsto X$ a transformation (bijection)
- ▶ $F : X \mapsto \mathbb{R}$ a function

Then 1-coboundary is just the difference

$$f := F \circ T - F.$$

These are “trivial cocycles, sums over T^n “telescope”.

$$\sum_0^n f \circ T^k = F \circ T^{n+1} - F$$

What are the 1-coboundaries for τ_α i.e. the Dehn twist round α ?

1-Coboundary

- ▶ Dehn twist action

$$\begin{aligned}\beta_{-1} &= \tau_\alpha^{-1}(\beta_0) = \alpha^{-1}\beta \in \pi_1(\Sigma) \\ \beta_{n+1} &:= \tau_\alpha(\beta_n) = \alpha.\beta_n\end{aligned}$$

$$\begin{array}{l} \tau_\alpha : \beta_{-1} \mapsto \beta_0 \qquad \mapsto \beta_1 \\ \text{Markoff triples} \quad (\alpha, \beta_0, \beta_{-1}) \mapsto (\alpha, \beta_1, \beta_0) \\ T : \quad \cdot \quad \mapsto (X, Y, Z^-) \mapsto (X, Z^+, Y) \end{array}$$

- ▶ Dehn twist action

$$T : (X, Y, Z) \mapsto ((X, XY - Z, Y).$$

- ▶ 1-coboundary $F := Z/Y = Z^-/Y$, $F \circ T = Y/Z^+$

$$\begin{aligned} F \circ T - T &= Y/Z^+ - Z^-/Y \\ &= \frac{Y^2 - (XY - Z)Z}{YZ} = \frac{Y^2 + Z^2 - XYZ}{YZ} = \frac{X^2}{YZ} \end{aligned}$$

1-Coboundary

Write the action as

$$T : (X, Y_n, Y_{n-1}) \mapsto (X, Y_{n+1}, Y_n)$$

$$\begin{pmatrix} Y_{n+1} \\ Y_n \end{pmatrix} := \begin{pmatrix} X & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y_n \\ Y_{n-1} \end{pmatrix}$$

Consider the cocycle $f = F \circ T - T = \frac{X^2}{YZ}$ and sum

$$\begin{aligned} \sum_{-\infty}^{\infty} f \circ T^n &= \lim_{n \rightarrow \infty} Y_{n+1}/Y_n - \lim_{n \rightarrow -\infty} Y_{n+1}/Y_n \\ &= \text{difference of the eigenvalues} \\ &= e^{\ell_\alpha/2} - e^{-\ell_\alpha/2} \\ &= 2 \sinh(\ell_\alpha/2) \end{aligned}$$

Bowditch's method: line integral in a graph

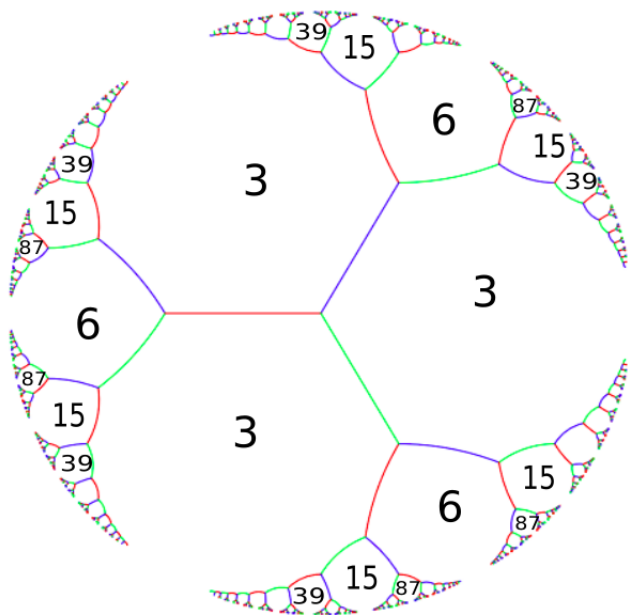
Let $\Gamma = (V, E)$ be the Bass-Serre tree embedded in the unit disc as before. Each complementary

Let $\Gamma^* = (E, E^*)$ where two edges are adjacent if they meet in a vertex :

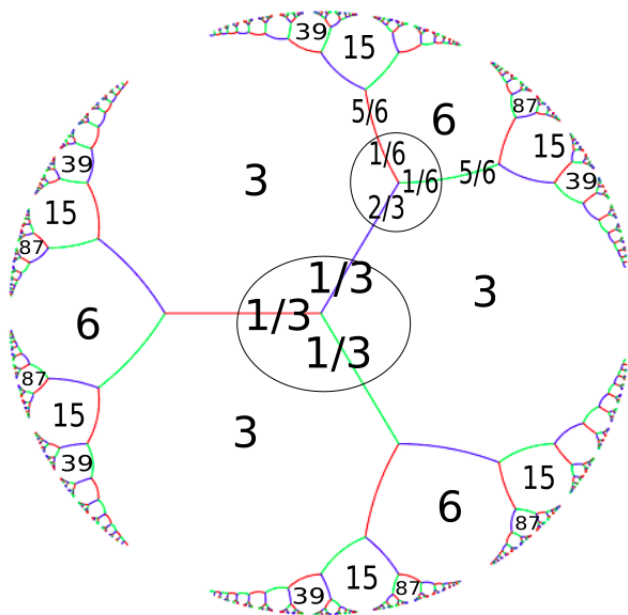
- ▶ not a tree there are 3 cycles.
- ▶ every vertex is order 4.
- ▶ mark the edges with values of the 1-coboundary
- ▶ round each cycle sum of the 1-coboundary is 1

$$\begin{aligned} X^2 + Y^2 + Z^2 &= XYZ \\ \frac{X}{YZ} + \frac{Y}{XZ} + \frac{Z}{XY} &= 1 \end{aligned}$$

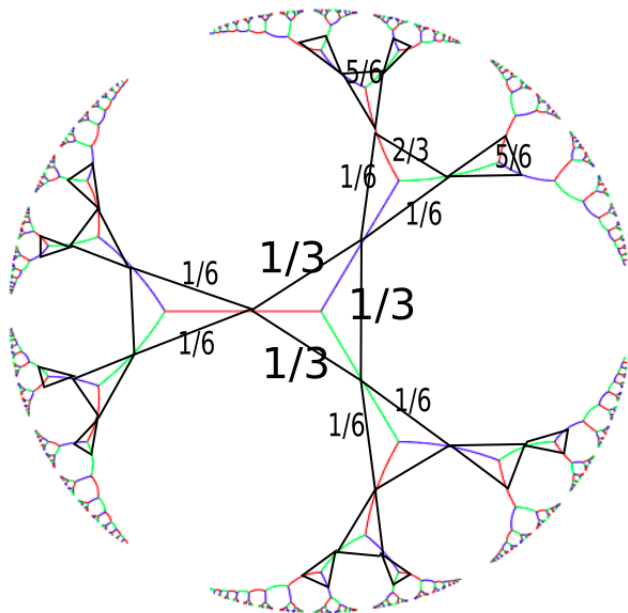
Labelled Γ^*



Labelled Γ^*



Labelled Γ^*

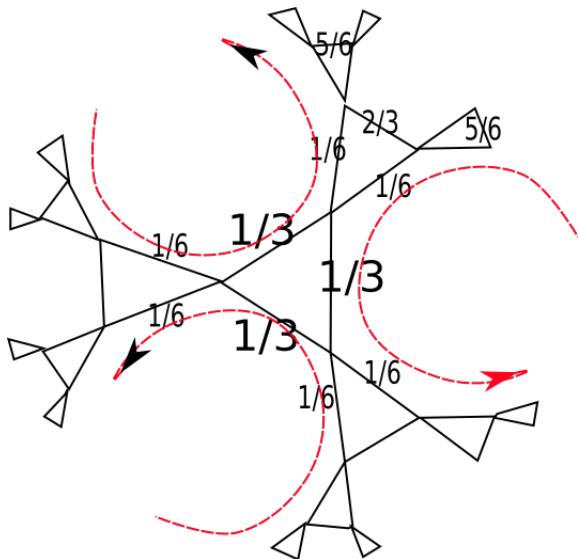


Bowditch's method: line integral in a graph

Let $\Gamma^* = (E, E^*)$

- ▶ mark the edges with values of the coboundary
 - ▶ round each cycle sum of the coboundary is 1
1. Take a connected subgraph with no vertices of valence 1. i.e. a union of n 3-cycles.
 2. Sum the coboundaries = $\#$ 3 cycles = n
 3. Separate the edges into **arcs** = connected unions of edges in the same complementary region of Γ
 4. Deal with the arcs in 2 ways
 - ▶ if the arc is a single edge then show that the coboundary value is very small
 - ▶ if the arc is long approximate the sum by
$$(1/X) \sum_{-\infty}^{\infty} f \circ T^n = 2\sinh(\ell_\alpha/2)/2\cosh(\ell_\alpha/2) = \tanh(\ell_\alpha/2)$$

Long and short



Bowditch's method: line integral in a graph

Let $\Gamma^* = (E, E^*)$

- ▶ n = sum of coboundaries over a subgraph
1. **arc** = connected unions of edges in the same complementary region of Γ
 2.
 - ▶ if the arc is a single edge then the coboundary value is very small
 - ▶ if the arc is long approximate the sum by
$$\frac{1}{X} \sum_{-\infty}^{\infty} f \circ T^n = 2\sinh(\ell_\alpha/2)/2\cosh(\ell_\alpha/2) = \tanh(\ell_\alpha/2)$$
 3. So for a finite number of closed simple geodesics in 1-1 correspondences with the log arcs :

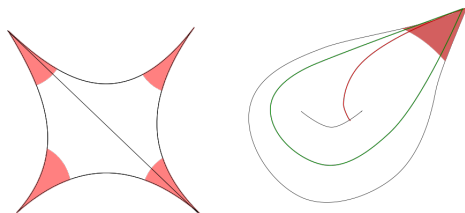
$$\sum_{\alpha} \tanh(\ell_\alpha/2) = n - o(1) \Rightarrow \sum_{\alpha} 1 - \tanh(\ell_\alpha/2) = 1 - o(1)$$

Moral

What Bowditch does is to sum over all possible values of the coboundary X/YZ

$$\frac{X}{YZ} + \frac{Y}{XZ} + \frac{Z}{XY} = 1$$

allows one to see that the sum is constant after rescaling.



Pink region has area 2. $\frac{X}{YZ}$, $\frac{Y}{XZ}$, $\frac{Z}{XY}$ are "angles" at the cusp.

Part V

Proof by differentiation

Differentiation

α, β closed oriented geodesics meet just once at x then

- ▶ In fact there are 3 points - the **Weierstrass points** such that ever closed simple geodesic passes through exactly 2 of the 3 points.
- ▶ $\sinh(\ell_\alpha/2) \sinh(\ell_\beta/2) \sin(\alpha\angle\beta) = \cosh(\ell_\gamma/4)$.

The sum of all the angles at a Weierstrass point ?

Let $G < \mathrm{PSL}(2, \mathbb{Z}) \simeq \mathcal{MCG}(\Sigma)$ be the maximal subgroup that “fixes Weierstrass points i.e.

$$x = \alpha \cap \beta = \phi(\alpha) \cap \phi(\beta), \forall \phi \in G.$$

Let J be the element $J(\alpha) = \beta, J(\beta) = -\alpha$ so that

$$\alpha\angle\beta + J(\alpha)\angle J(\beta) = \alpha\angle\beta + \beta\angle(-\alpha) = \pi.$$

$$\begin{aligned} \sum_G \phi(\alpha)\angle\phi(\beta) &= \sum_{J \setminus G} \phi(\alpha)\angle\phi(\beta) + \sum_{J \setminus G} \phi(J(\alpha))\angle\phi(J(\beta)) \\ &= \infty \times \pi \end{aligned}$$

Differentiation

Want to show that the derivative of $\sum_G \phi(\alpha)\angle\phi(\beta)$ is constant using

$$\alpha\angle\beta + \beta\angle(-\alpha) = \pi \Rightarrow d\alpha\angle\beta + d\beta\angle(-\alpha) = 0$$

so we have to show that $\sum_G d\phi(\alpha)\angle\phi(\beta)$ is **absolutely convergent**.

$$\sin(\alpha\angle\beta) = \frac{\cosh(l_\gamma/4)}{\sinh(l_\alpha/2)\sinh(l_\beta/2)}.$$

so that

$$d(\alpha\angle\beta) = A(l_\alpha, l_\beta)dl_\alpha + B(l_\alpha, l_\beta)dl_\beta$$

Differentiation

Theorem (Kerckhoff, Wolpert)

$$dl_\gamma \cdot \frac{\partial}{\partial \tau_\alpha} = \sum_{x \in \alpha \cap \gamma} \cos \theta_x$$

where

- ▶ dl_γ is the variation of l_γ
- ▶ $\partial(\cdot)/\partial \tau_\alpha$ is the vector field that generates the Fenchel-Nielsen twist deformation τ_α along α
- ▶ θ_x is the angle between α and γ at the intersection point x

Corollary (Remark?)

$$dl_\alpha \cdot \frac{\partial}{\partial \tau_\alpha} = \sum_{x \in \emptyset} = 0,$$

i.e. $\frac{\partial}{\partial \tau_\alpha}$ is a tangent vector to the level sets of l_α .

Differentiation: Transversality

- ▶ $\frac{\partial}{\partial \tau_\alpha}$ is a tangent vector to the level sets of l_α .
- ▶ Let α, β a pair of closed simple geodesics that meet in a single point.

$$dl_\alpha \cdot \frac{\partial}{\partial \tau_\alpha} = 0, \quad dl_\gamma \cdot \frac{\partial}{\partial \tau_\alpha} = \cos \theta_x \neq 0$$

- ▶ The vector fields

$$\frac{\partial}{\partial \tau_\alpha}, \frac{\partial}{\partial \tau_\beta}$$

are a basis for the tangent bundle on an open dense set of $\mathcal{T}(\Sigma)$ namely $\{X \in \mathcal{T}(\Sigma), \cos \theta_x \neq 0\}$.

Corollary (Trivial Bound)

$$\left| dl_\gamma \cdot \frac{\partial}{\partial \tau_\alpha} \right| \leq i(\alpha, \gamma) := \#(\alpha \cap \gamma)$$

Differentiation: Absolute convergence

Lemma

Let α, β a pair of closed simple geodesics that meet in a single point. Then

$$\|dl_\gamma\| := \left| dl_\gamma \cdot \frac{\partial}{\partial \tau_\alpha} \right| + \left| dl_\gamma \cdot \frac{\partial}{\partial \tau_\beta} \right|$$

defines a norm on $\{x, \cos \theta_x \neq 0\}$.

Of course one has the bound :

$$\|dl_\gamma\| \leq i(\alpha, \gamma) + i(\beta, \gamma).$$

Differentiation: Absolute convergence

Using the trivial bound and the formula for $\alpha\angle\beta$:

Theorem (M.)

The derivatives of the angle sum converge absolutely.

Lemma

$\alpha\angle\beta$ is a coboundary for τ_γ .

Proof.

Complete γ to be a Markoff triple of geodesics α, β, γ so that

$$\beta = \gamma\alpha \in \pi_1(\Sigma) \Rightarrow \beta = \tau_\gamma(\alpha).$$

Take the shortest arc I between $x = \alpha \cap \beta$ and define a function F on simple arcs from x to γ .

$$\alpha\angle\beta = I\angle\beta - I\angle\alpha = F \circ \tau_\gamma - F.$$



An identity

Using the cocycle and renormalising correctly we prove :

Theorem (M.)

$$\sum_{\alpha} \arctan \left(\frac{\cosh(\ell_{\delta}/4)}{\sinh(\ell_{\alpha}/2)} \right) = \frac{3\pi}{2}$$

We can split this into 3 identities using the Weierstrass points:

$$\sum_{\alpha} \arctan \left(\frac{\cosh(\ell_{\delta}/4)}{\sinh(\ell_{\alpha}/2)} \right) = \frac{\pi}{2}$$

One can find the value of the series by letting $\ell_{\alpha} \rightarrow 0$:

- ▶ the biggest term $\rightarrow \arctan(\infty) = \pi/2$
- ▶ all other terms $\rightarrow 0$.

but there is a delicate issue of convergence...

An identity from the sum of all the angles

Final remarks : 3 methods for proving identities

- ▶ Geometric
- ▶ Integral
- ▶ Differential

It may be possible to prove relations between sums of lengths by realising Γ^* as a path in Teichmueller space and “pulling back” 1-forms on Teichmueller space to Γ^* .

