The reduced Dijkgraaf-Witten invariant of twist knots in the Bloch group of $\mathbb{F}_p$

Hiroaki KARUO
RIMS, Kyoto Univ.

Intro.

M: ori. closed 3-mfd. G: discrete gp.

$\pi_1(M) \xrightarrow{\rho} G$

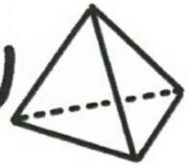
$\rightsquigarrow$

$H_3(M) \xrightarrow{\rho_*} H_3(BG) = H_3(G)$

$[M] \longmapsto \rho_*[M]$ (DW inv.)

$\rho_*[M] \in H_3(G)$ $\xrightarrow[\text{3-cocycle}]{\text{[D-W '90]}}$ Dijkgraaf-Witten inv.

3-cocycle of G

[Wakui '92] $\{ M = \bigcup$  $\}$ $\rightsquigarrow$ Dijkgraaf-Witten inv.

p : prime

$\mathbb{F}_p$: fin. field of order p

$SL_2 F \curvearrowright \mathbb{P}^1(F)$

$F = \mathbb{C}$ $\rightsquigarrow$ [Neumann '04]

$F = \mathbb{F}_p$ $\rightsquigarrow$ [Hutchinson '13]

$H_3(SL_2\mathbb{C}) \to \check{\mathcal{B}}(\mathbb{C})$ [Neumann '04]

$\rho_*[M] \longmapsto$ cplx. hyp. vol.

$H_3(SL_2\mathbb{F}_p) \to \check{\mathcal{B}}(\mathbb{F}_p)$

$\rho_*[M] \longmapsto$ reduced DW inv. $\widehat{DW}(M, \mathbb{F}_p)$

Main thm's

Calculate reduced DW inv. for $p = 7, 11, 13$ & $S^3 \setminus$ 

Contents

§1. Bloch gp., Dehn filling

$$\mathbb{Z}(\mathbb{F}_p^{\times}\setminus\{1\}) \xrightarrow{/①} \underset{\cup}{\mathcal{P}(\mathbb{F}_p)} \xrightarrow{/②} \underset{\cup}{\check{\mathcal{P}}(\mathbb{F}_p)}$$

$$\mathrm{Ker}(\mathcal{P}(\mathbb{F}_p) \to \Lambda^2 \mathbb{F}_p^{\times}) = \mathcal{B}(\mathbb{F}_p) \to \check{\mathcal{B}}(\mathbb{F}_p)$$

$$z \mapsto z \wedge (1-z)$$

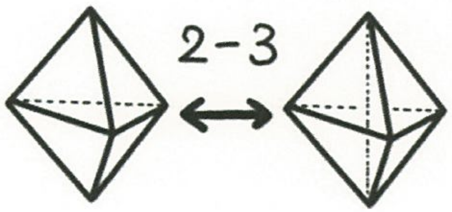
Bloch gp.

Ex. $\check{\mathcal{B}}(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z}$

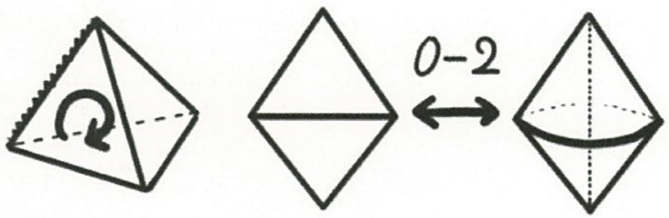
$\check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\}$

$\check{\mathcal{B}}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}$

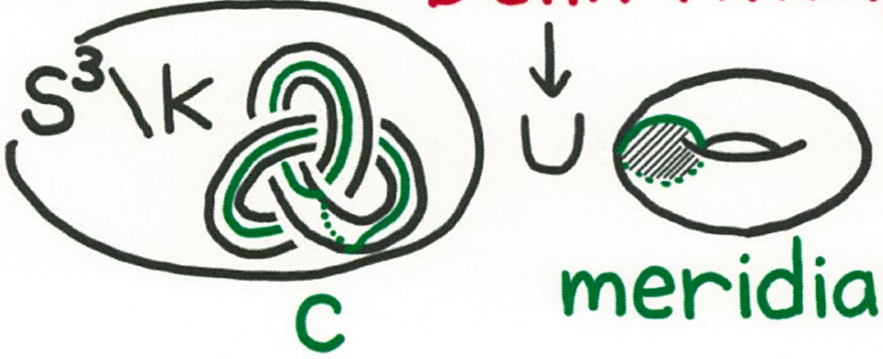
① $[x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+\left[\frac{1-x}{1-y}\right]=0$ $(x, y \in \mathbb{F}_p^{\times}\setminus\{1\})$



② $[x]=\left[1-\frac{1}{x}\right]=\left[\frac{1}{1-x}\right]=-\left[\frac{1}{x}\right]$



K: knot — **Dehn filling** along c



$= M_c(K)$: ori. closed 3-mfd.

reduced DW inv. (closed)

M : ori. closed 3-mfd.

$$SL_2\mathbb{F}_p \curvearrowright \mathbb{P}^1(\mathbb{F}_p)$$

$$(g_0, g_1, \cdots, g_n) \longmapsto (g_0 z, g_1 z, \cdots, g_n z)$$

$$\begin{array}{ccc}
\pi_1(M) \xrightarrow{\ \rho\ } SL_2\mathbb{F}_p & \qquad C_*(SL_2\mathbb{F}_p) & \longrightarrow \; C_*\left(\begin{array}{c}\text{chain complex of} \\ (n+1)\text{-tuple of } \mathbb{P}^1(\mathbb{F}_p)\end{array}\right) \\
\Big\downarrow & \Big\downarrow H_3(\cdot) & \Big\downarrow H_3(\cdot) \\
H_3(M) \xrightarrow{\ \rho_*\ } H_3(BSL_2\mathbb{F}_p) & = H_3(SL_2\mathbb{F}_p) & \longrightarrow \; \check{\mathcal{B}}(\mathbb{F}_p) \\
\Psi \qquad\qquad \Psi & \in & \Psi \\
[M] \longmapsto \rho_*[M] & \longmapsto & \widehat{DW}(M, \rho)
\end{array}$$

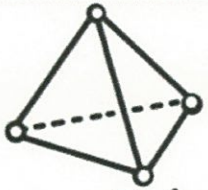
$\rho_*[M]$: DW inv.

$$\widehat{DW}(M, \rho) \;\rightsquigarrow\; \sum_\rho$$

$$\widehat{DW}(M, \mathbb{F}_p) \overset{\text{def}}{=} \sum_\rho \widehat{DW}(M, \rho) \in \mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_p)]$$

reduced DW inv.

reduced DW inv. ($S^3 \setminus$ knot)

$M = \bigcup$  ← Extend ← ($M = \bigcup$  closed 3-mfd.)

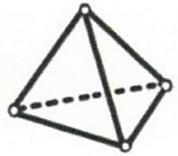
$\| $

$S^3 \setminus K$ 3-mfd. with cusp

$$\pi_1(S^3 \setminus K) \xrightarrow{\rho} SL_2\mathbb{F}_p$$

$c \in \operatorname{Ker}\rho$ ↓ , ↗ ρ

$$\pi_1(M_c(K))$$

$\sum$ [modulus of ]

$\|$

reduced DW inv. $\widehat{DW}(K, \mathbb{F}_p) \overset{\text{def}}{=} \sum_{\rho:\text{para}} \widehat{DW}(M_c(K), \rho)$ ← does not depend on c

$\rho : \pi_1(S^3 \setminus K) \to SL_2\mathbb{F}_p$: parabolic rep.

$\overset{\text{def}}{\Leftrightarrow} \rho(\text{meridian}) = Q^{-1}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} Q \quad (Q \in SL_2\overline{\mathbb{F}_p},\ * \in \mathbb{F}_p^\times)$

§2. Thm's ($p = 7, 11$)

T_n =  : n-twist knot, $|SL_2\mathbb{F}_p| = p(p^2-1)$

Thm1. $\widehat{DW}(T_n, \mathbb{F}_7) = A_n + B_n \in \mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_7)] \cong \mathbb{Z}\langle t \mid t^2 = 1\rangle$, where $\check{\mathcal{B}}(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z}$

$$A_n = \begin{cases} t & \text{if } n \equiv 2, 3 \bmod 6 \\ 0 & \text{otherwise} \end{cases} \qquad B_n = \begin{cases} t^{(n-1)/8} & \text{if } n \equiv 1 \bmod 8 \\ t^{(n+6)/8} & \text{if } n \equiv 2 \bmod 8 \\ t^{(n-5)/8} & \text{if } n \equiv 5 \bmod 8 \\ t^{(n+2)/8} & \text{if } n \equiv 6 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

Thm2. $\widehat{DW}(T_n, \mathbb{F}_{11}) = A_n + B_n \in \mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_{11})] = \mathbb{Z}$, where $\check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\}$

$$A_n = \begin{cases} 1 & \text{if } n \equiv 1, 3 \bmod 5 \\ 0 & \text{otherwise} \end{cases} \qquad B_n = \begin{cases} 1 & \text{if } n \equiv 4, 5, 6, 7 \bmod 12 \\ 0 & \text{otherwise} \end{cases}$$

Rmk $\check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\} \rightsquigarrow \widehat{DW}(T_n, \mathbb{F}_{11}) = \#\left\{\begin{matrix}\text{parabolic rep.'s} \\ \pi_1(S^3 \setminus T_n) \to SL_2\mathbb{F}_{11}\end{matrix}\right\} \Big/ \text{conj.}$

Thm (p = 13)

$$\check{B}(\mathbb{F}_{13}) \ni [2]=[7]=[12],\ -[3]=-[5]=-[6]=[8]=[9]=[11],\ [4]=-[10]$$

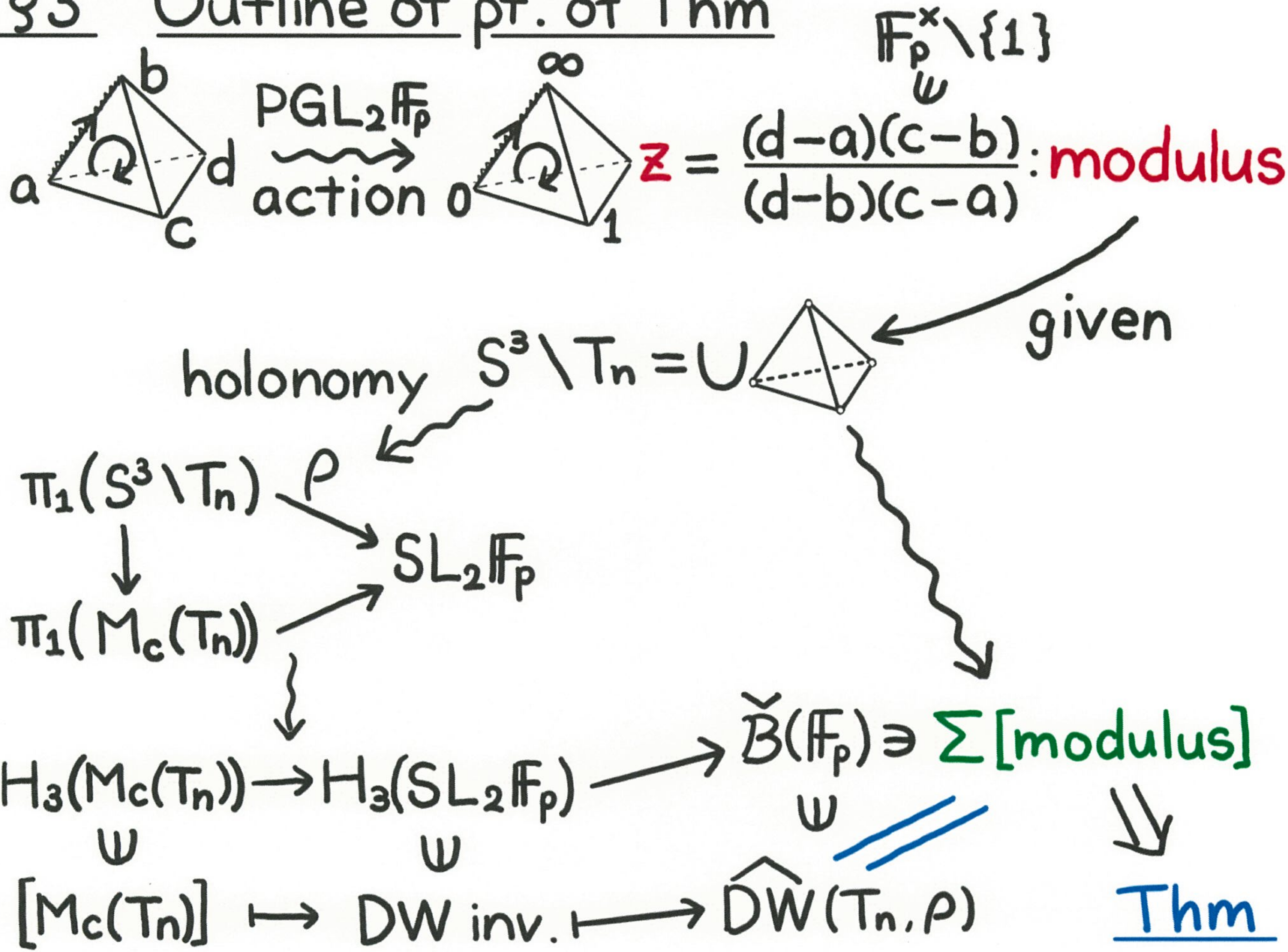
$$\begin{array}{cccc} \check{B}(\mathbb{F}_{13}) \ni & [2]=[7]=[12] & -[3]=-[5]=-[6]=[8]=[9]=[11] & [4]=-[10] \\ \cong & \downarrow & \downarrow & \downarrow \\ \mathbb{Z}/7\mathbb{Z} \ni & 0 & 1 & 2 \end{array}$$

Thm3. $\widehat{DW}(T_n, \mathbb{F}_{13}) = A_n + B_n \in \mathbb{Z}[\check{B}(\mathbb{F}_{13})] \cong \mathbb{Z}\langle t \mid t^7 = 1 \rangle$, where $\check{B}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}$

$$A_n = \begin{cases} t & \text{if } n \equiv 10 \mod 13 \\ t^4 & \text{if } n \equiv 2 \mod 13 \\ 0 & \text{otherwise} \end{cases}$$

$$B_n = \begin{cases} t^{(n-1)/7} & \text{if } n \equiv 1 \mod 7 \\ t^{(2n+17)/7} & \text{if } n \equiv 2 \mod 7 \\ t^{(2n-15)/7} & \text{if } n \equiv 4 \mod 7 \\ t^{(n+2)/7} & \text{if } n \equiv 5 \mod 7 \\ 0 & \text{otherwise} \end{cases}$$

§3 Outline of pf. of Thm



reduced DW inv. is summation of moduli

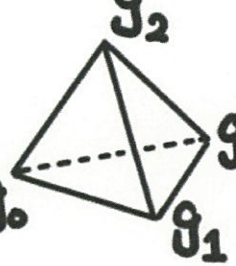
[Hutchinson'13]

$$H_3(M) \longrightarrow H_3(SL_2\mathbb{F}_p) \longrightarrow \check{\mathcal{B}}(\mathbb{F}_p)$$

$$H_3(M) \longrightarrow H_3(PGL_2\mathbb{F}_p), \quad H_3(SL_2\mathbb{F}_p) \longrightarrow H_3(PGL_2\mathbb{F}_p)$$

$$H_3(PGL_2\mathbb{F}_p) \longrightarrow H_3(\widehat{C}_*) \longrightarrow \check{\mathcal{B}}(\mathbb{F}_p), \quad H_3(PGL_2\mathbb{F}_p) \longrightarrow \check{\mathcal{B}}(\mathbb{F}_p)$$

(all squares commute ↻)

$\sum$  $= \sum (g_0, g_1, g_2, g_3) \overset{w}{\longmapsto} \sum (g_0 w, \cdots, g_3 w) \longmapsto \sum [g_0 w, \cdots, g_3 w]$

$w \in \mathbb{P}^1(\mathbb{F}_p)$

does not depend on w

$\sum [g_0 w, \cdots, g_3 w] = \sum[\text{modulus}]$

$\widehat{C}_*$ (chain complex of (n+1)-tuple of pt's of $\mathbb{P}^1(\mathbb{F}_p)$) ← maybe duplicate

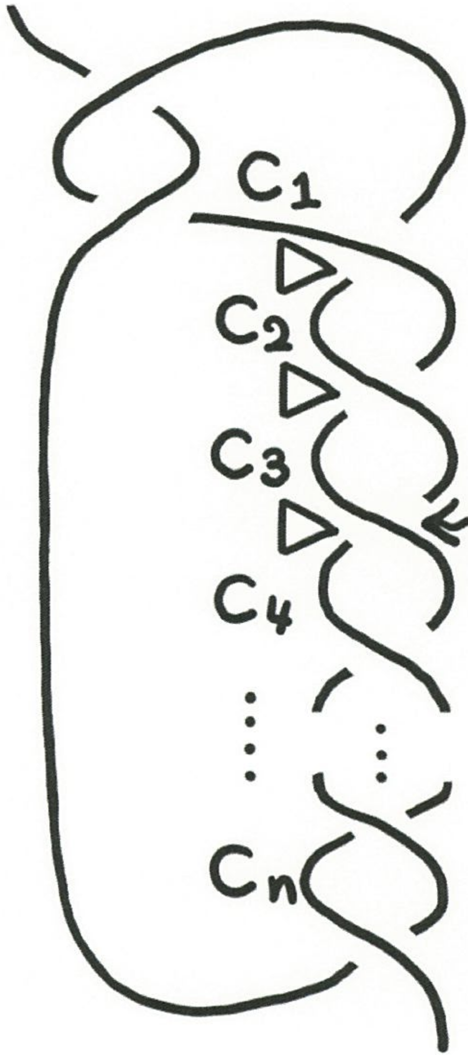
$$[a,b,c,d] = \begin{cases} \dfrac{(d-a)(c-b)}{(d-b)(c-a)} & \text{if not duplicate} \\ 0 & \text{otherwise} \end{cases}$$

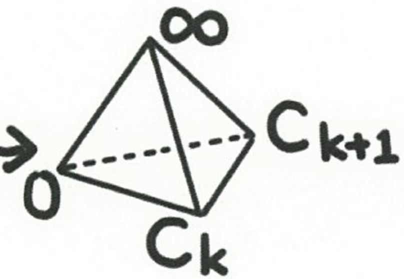
How to calculate reduced DW inv.

hyperbolicity eq.'s

$\rightsquigarrow$

$$c_k = 1 - \frac{c_{k-1}}{c_{k-2}} + c_{k-1}$$





periodic

order = divisor of

$$|SL_2\mathbb{F}_p| = p(p+1)(p-1)$$

$$\text{modulus} = \left[\frac{c_{k+1}}{c_k}\right] \in \check{\mathcal{P}}(\mathbb{F}_p)$$

$$\widehat{DW}(T_n, \rho_{c_1}) = \circledast + \sum \left[\frac{c_{k+1}}{c_k}\right] \in \check{\mathcal{B}}(\mathbb{F}_p)$$

$$\widehat{DW}(T_n, \mathbb{F}_p) = \sum_{c_1} \widehat{DW}(T_n, \rho_{c_1}) \in \mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_p)]$$

$\Downarrow$

Thm

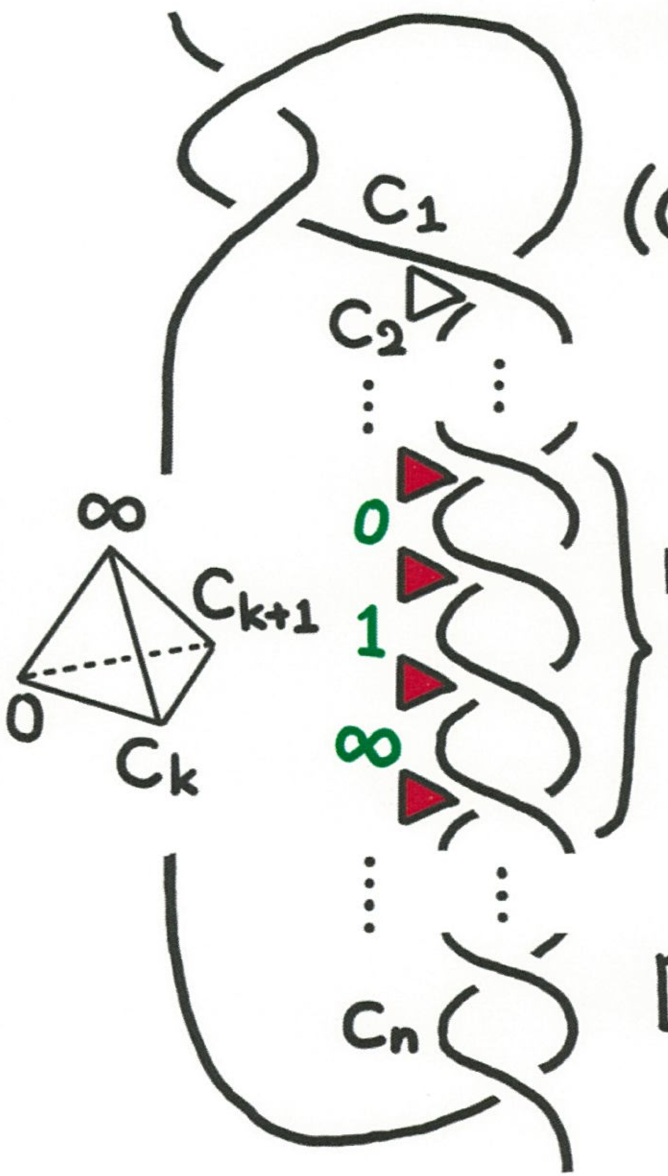
How to calculate reduced DW inv.(collapsed)

hyperbolicity eq.'s

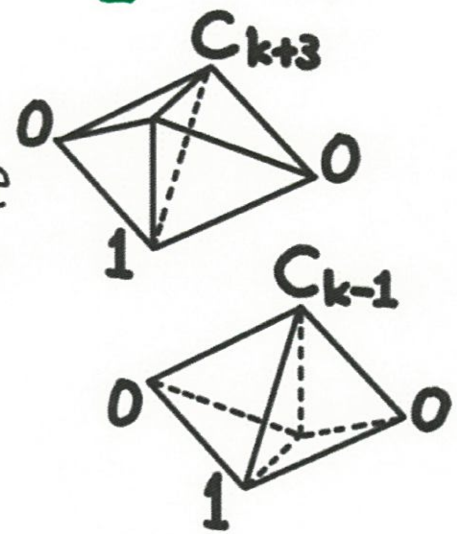
⇝

$(c_1,\cdots,c_k,c_{k+1},c_{k+2},\cdots,c_n)\left(\begin{matrix}c_{k-1}\\ c_{k+3}\end{matrix}\in\mathbb{F}_p^{\times}\setminus\{1\}\right)$

$c_k = 0,\quad c_{k+1} = 1,\quad c_{k+2} = \infty$



replace ⇝



$\Sigma\,[\text{modulus}]=0$

$\widehat{DW}(T_n,\rho_{c_1}) = \oslash + \Sigma\,[\text{modulus}] \in \check{\mathcal{B}}(\mathbb{F}_p)$

uncollapsed

⇓

Thm