Non-Semisimple TQFTs & Quantum Groups

(Joint Work with Nathan Geer and Bertrand Patureau)

Marco De Renzi

早稲田大学

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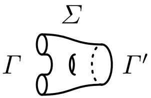
Topological Quantum Field Theories

Definition (TQFT)

Symmetric monoidal functor V from a category of cobordisms to a category of vector spaces

Idea

- Closed $d-1$-manifold $\mapsto$ Vector space
- d-Manifold / homeomorphism relative to $\partial \mapsto$ Linear map
- Disjoint union $\mapsto$ Tensor product
- Gluing along $\partial \mapsto$ Composition



$$\overset{V}{\longmapsto} \qquad V(\Gamma) \xrightarrow{\;V(\Sigma)\;} V(\Gamma')$$

Why TQFTs?

Selling Points

- Topological invariants with good locality properties, i.e. computable by cut-and-paste methods
- Deep applications outside of quantum topology, e.g. representations of mapping class groups

Non-Semisimple Constructions

- TQFT constructions usually have algebraic flavor
- Typical ingredients include quantum groups
- Classical approaches require semisimplicity

Warning

Quantum groups occurring in nature are not semisimple, so they have to undergo a quotient process which sacrifices information

Idea

Come up with constructions which work in non-semisimple settings

CGP Invariants

Algebraic Ingredient

U^H unrolled $\mathfrak{sl}_2$ at $q = e^{\frac{2\pi i}{r}}$ with $r \geqslant 3$ integer

Topological Invariant (Costantino-Geer-Patureau)

$\mathrm{N}_r(M, T, \omega) \in \mathbb{C}$ defined via surgery for admissible triples

- M closed oriented 3-manifold
- $T \subset M$ framed link colored with U^H-modules
- $\omega \in H^1(M \smallsetminus T; \mathbb{C}/2\mathbb{Z})$ compatible cohomology class

Admissibility

- Either T admits a color in $\mathrm{Proj}(\mathscr{C}^H)$
- Or ω admits an evaluation in $\mathbb{C}/2\mathbb{Z} \smallsetminus \mathbb{Z}/2\mathbb{Z}$

CGP TQFTs

Theorem (Blanchet-Costantino-Geer-Patureau,D)

N_r *extends to a* $\mathbb{Z}$*-graded TQFT* $\mathbb{V}_r$ *for every integer* $r \geqslant 3$

Highlights

- Akutsu-Deguchi-Ohtsuki link invariant is contained in N_r
- Abelian Reidemeister torsion is contained in N_4
- Non-separating Dehn twists have infinite order under $\mathbb{V}_r$

Renormalized Hennings Invariants

Algebraic Ingredient

$\bar{U}$ small/restricted $\mathfrak{sl}_2$ at $q = e^{\frac{2\pi i}{r}}$ with $r \geqslant 3$ odd/even integer

Topological Invariant (D-Geer-Patureau)

$\mathrm{H}'_r(M,T) \in \mathbb{C}$ defined via surgery for admissible pairs

- M closed oriented 3-manifold
- $T \subset M$ framed link colored with $\bar{U}$-modules

Admissibility

T admits a color in $\mathrm{Proj}(\bar{\mathscr{C}})$

Renormalized Hennings TQFTs

Theorem (D-Geer-Patureau)

H'_r *extends to a TQFT* V_r *for every odd integer* $r \geqslant 3$

Highlights

- Constructions are less technical
- Generalized Kashaev invariant of Murakami is contained in H'_r
- Non-separating Dehn twists have infinite order under $\mathbb{V}_r$

Main Result

- $\bar{\mathscr{C}}$ category of finite-dimensional $\bar{U}$-modules
- $\mathscr{C}^H$ category of finite-dimensional weight U^H-modules
- $\mathscr{C}^H \cong \bigoplus_{\bar{\alpha} \in \mathbb{C}/2\mathbb{Z}} \mathscr{C}^H_{\bar{\alpha}}$ with respect to weights
- $\Phi : \mathscr{C}^H_{\bar{0}} \to \bar{\mathscr{C}}$ forgetful functor

Theorem (D-Geer-Patureau)

$\mathrm{N}_r(M, T, 0) = \mathrm{H}'_r(M, \Phi(T))$ *for every odd integer* $r \geqslant 3$

- Proof uses TQFT extensions
- Result can be generalized to every simple Lie algebra

Quantum Groups of $\mathfrak{sl}_2$

Fix odd integer $r \geqslant 3$

Small quantum group $\bar{U}$

- Generators: E, F, K, K^{-1}
- Relations: $KK^{-1} = K^{-1}K = 1$,
  $KEK^{-1} = q^2E$, $\quad KFK^{-1} = q^{-2}F$, $\quad [E, F] = \frac{K-K^{-1}}{q-q^{-1}}$,
  $K^r = 1$, $\quad E^r = F^r = 0$

Unrolled quantum group U^H

- Generators: E, F, K, K^{-1}, H
- Relations: $KK^{-1} = K^{-1}K = 1$,
  $KEK^{-1} = q^2E$, $\quad KFK^{-1} = q^{-2}F$, $\quad [E, F] = \frac{K-K^{-1}}{q-q^{-1}}$,
  $[H, K] = 0$, $\quad [H, E] = 2E$, $\quad [H, F] = -2F$, $\quad E^r = F^r = 0$

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Small Quantum Group

$\bar{\mathscr{C}}$ category of finite-dimensional $\bar{U}$-modules

Rigidity

Every $V \in \bar{\mathscr{C}}$ determines

$$
\begin{array}{rrcl}
\overleftarrow{\mathrm{ev}}_V : & V^* \otimes V & \to & \mathbb{C} \\
& f \otimes v & \mapsto & f(v) \\
\overrightarrow{\mathrm{ev}}_V : & V \otimes V^* & \to & \mathbb{C} \\
& v \otimes f & \mapsto & f(g \cdot v)
\end{array}
\qquad
\begin{array}{rrcl}
\overleftarrow{\mathrm{coev}}_V : & \mathbb{C} & \to & V \otimes V^* \\
& 1 & \mapsto & \sum_i v_i \otimes f_i \\
\overrightarrow{\mathrm{coev}}_V : & \mathbb{C} & \to & V^* \otimes V \\
& 1 & \mapsto & \sum_i f_i \otimes g^{-1} \cdot v_i
\end{array}
$$

with $\{v_i\}$ basis of V, $\{f_i\}$ dual basis, and $g := K \in \bar{U}$

Small Quantum Group

$\bar{\mathscr{C}}$ category of finite-dimensional $\bar{U}$-modules

Braiding

All $V, V' \in \bar{\mathscr{C}}$ determine

$$\begin{array}{rccc} c_{V,V'}: & V \otimes V' & \to & V' \otimes V \\ & v \otimes v' & \mapsto & \tau(R \cdot v \otimes v') \end{array}$$

with $\tau(v \otimes v') := v' \otimes v$ for all $v \in V$, $v' \in V'$, and

$$R := \frac{1}{r} \sum_{a,b,c=0}^{r-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2} - 2bc} K^b E^a \otimes K^c F^a \in \bar{U} \otimes \bar{U}$$

$$\{a\} := q^a - q^{-a}, \quad [a] := \frac{\{a\}}{\{1\}}, \quad [a]! := [a][a-1] \cdots [1] \quad \forall\, a \in \mathbb{N}$$

Unrolled Quantum Group

$\mathscr{C}^H$ category of finite-dimensional weight U^H-modules

Weight Module

U^H-module V with H diagonalizable and $K = q^H$, meaning

$$H \cdot v = \alpha v \quad \Rightarrow \quad K \cdot v = q^\alpha v \quad \forall\, v \in V$$

$$q^\alpha := e^{\frac{2\alpha\pi i}{r}} \quad \forall\, \alpha \in \mathbb{C}$$

- $\mathscr{C}^H_{\bar{\alpha}}$ full subcategory of $\mathscr{C}^H$ with weights in $\bar{\alpha} \in \mathbb{C}/2\mathbb{Z}$
- $\Phi : \mathscr{C}^H_{\bar{0}} \to \bar{\mathscr{C}}$ forgets about H

Unrolled Quantum Group

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with $\{v_i\}$ basis of V, $\{f_i\}$ dual basis, and $g := K^{-r+1} \in U^H$

Unrolled Quantum Group

$\mathscr{C}^H$ category of finite-dimensional weight U^H-modules

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All $V, V' \in \mathscr{C}^H$ determine

$$\begin{array}{rccc} c_{V,V'} : & V \otimes V' & \to & V' \otimes V \\ & v \otimes v' & \mapsto & \tau(R(v \otimes v')) \end{array}$$

with $\tau(v \otimes v') := v' \otimes v$ for all $v \in V$, $v' \in V'$, and

$$R := q^{\frac{H \otimes H}{2}} \sum_{a=0}^{r-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2}} E^a \otimes F^a \in \mathrm{End}_{\mathscr{C}^H}(V \otimes V')$$

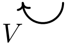
$$H \cdot v = \alpha v,\ H \cdot v' = \alpha' v' \ \Rightarrow\ q^{\frac{H \otimes H}{2}} \cdot v \otimes v' := q^{\frac{\alpha\alpha'}{2}} v \otimes v' \ \forall\, v \in V,\ v' \in V'$$

Reshetikhin-Turaev Functors

- $\mathscr{C}$ ribbon category
- $\mathscr{R}_{\mathscr{C}}$ category of $\mathscr{C}$-colored ribbon graphs

There exists $F_{\mathscr{C}} : \mathscr{R}_{\mathscr{C}} \to \mathscr{C}$ monoidal functor

$\mapsto \overleftarrow{\mathrm{ev}}_V$



$\mapsto \overleftarrow{\mathrm{coev}}_V$

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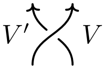
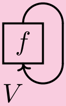


$\mapsto c_{V,V'}$

Problem with Non-Semisimple Categories

Warning

If $\mathscr{C}$ is non-semisimple then for all $V \in \mathrm{Proj}(\mathscr{C})$ and $f \in \mathrm{End}_{\mathscr{C}}(V)$



$$\doteq \ 0$$

Idea

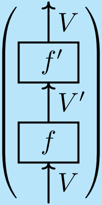
Replace trace with non-degenerate operation having same behavior

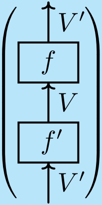
Projective Traces

$\mathscr{C}$ ribbon $\mathbb{k}$-linear category

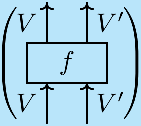
Definition (Trace on $\mathrm{Proj}(\mathscr{C})$)

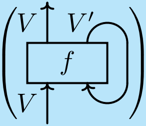
$\mathrm{t} := \{\mathrm{t}_V : \mathrm{End}_{\mathscr{C}}(V) \to \mathbb{k} \mid V \in \mathrm{Proj}(\mathscr{C})\}$

- t_V


$= \mathrm{t}_{V'}$


$\forall\, V, V' \in \mathrm{Proj}(\mathscr{C})$
$\forall\, f \in \mathrm{Hom}_{\mathscr{C}}(V, V')$
$\forall\, f' \in \mathrm{Hom}_{\mathscr{C}}(V', V)$

- $\mathrm{t}_{V \otimes V'}$


$= \mathrm{t}_V$


$\forall\, V \in \mathrm{Proj}(\mathscr{C})$
$\forall\, V' \in \mathscr{C}$
$\forall\, f \in \mathrm{End}_{\mathscr{C}}(V \otimes V')$

Existence and Non-Degeneracy of Projective Traces

Definition (Non-Degeneracy of Trace t on $\mathrm{Proj}(\mathscr{C})$)

The pairing $\mathrm{t}_V(\cdot \circ \cdot) : \mathrm{Hom}_{\mathscr{C}}(V', V) \otimes \mathrm{Hom}_{\mathscr{C}}(V, V')$ is non-degenerate for all $V \in \mathrm{Proj}(\mathscr{C}), V' \in \mathscr{C}$

Theorem (Geer-Patureau-Virelizier)

Up to scalar, there exists a unique trace t^H *on* $\mathrm{Proj}(\mathscr{C}^H)$*, and furthermore* t^H *is non-degenerate*

Theorem (Beliakova-Blanchet-Gainutdinov)

Up to scalar, there exists a unique trace $\bar{\mathrm{t}}$ *on* $\mathrm{Proj}(\bar{\mathscr{C}})$*, and furthermore* $\bar{\mathrm{t}}$ *is non-degenerate*

CGP Theory

Strategy

Work in the semisimple part of $\mathscr{C}^H$

- $\alpha \in \mathbb{C} \smallsetminus \mathbb{Z}$
- V_α simple projective U^H-module of highest weight α
- $\mathscr{C}^H_{\bar{\alpha}}$ semisimple and dominated by $\{V_{\alpha+2j} \mid j \in \mathbb{Z}\}$
- $\mathrm{d}^H(V_{\alpha+2j}) := \frac{r\{\alpha+2j+1\}}{\{r\alpha\}}$ where $\{\alpha\} := q^\alpha - q^{-\alpha} \; \forall \, \alpha \in \mathbb{C}$
- $\Omega_\alpha := \sum_{j=0}^{r-1} \mathrm{d}^H(V_{\alpha+2j}) V_{\alpha+2j}$ Kirby color

Sketch of CGP Construction

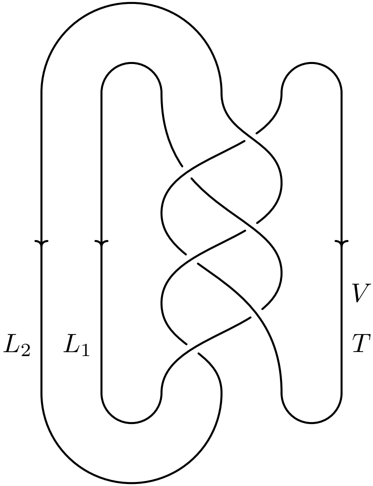
- (M, T, ω) admissible connected triple
- $L = L_1 \cup \ldots \cup L_\ell \subset S^3$ oriented surgery presentation of M of signature $\sigma(L)$
- L_ω obtained from L by labeling every component L_i with Ω_{α_i} such that $\langle \omega, m_i \rangle = \bar{\alpha}_i$ for a positive meridian m_i of L_i
- $L'_\omega \cup T'$ obtained from $L_\omega \cup T$ by cutting an edge of color $V \in \mathrm{Proj}(\mathscr{C}^H)$

CGP Invariant

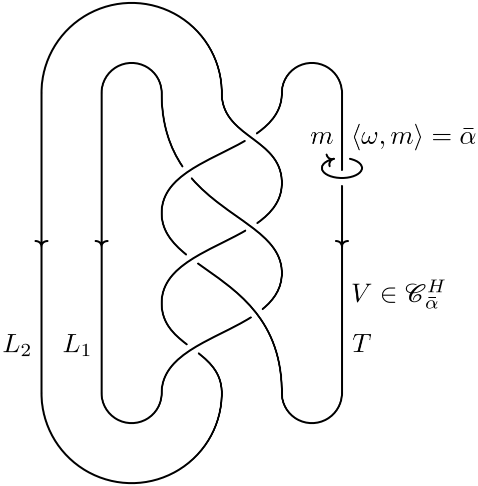
$$\mathrm{N}_r(M, T, \omega) := \mathscr{D}^{-1-\ell} \delta^{-\sigma(L)} \mathrm{t}_V^H (F_{\mathscr{C}^H}(L'_\omega \cup T'))$$

$$\mathscr{D} := r^{\frac{3}{2}} \qquad \delta := i^{-\frac{r-1}{2}} q^{\frac{r-3}{2}}$$

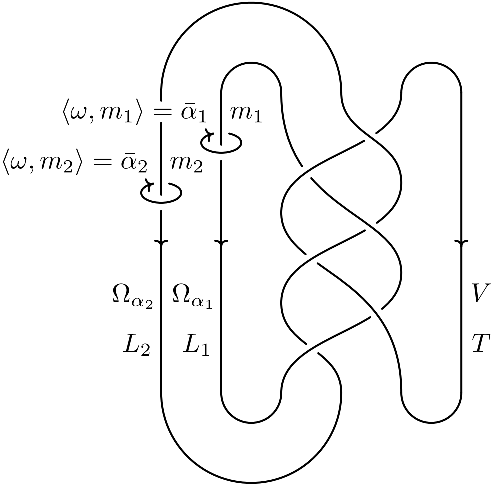
Example



Example

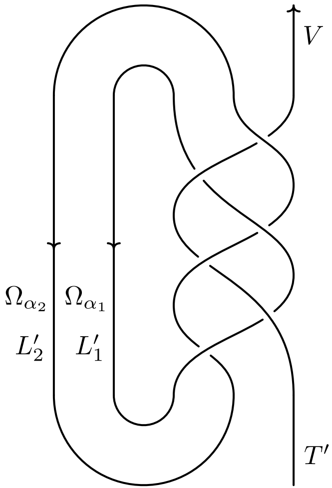

m
$\langle \omega, m \rangle = \bar{\alpha}$
$V \in \mathscr{C}^H_{\bar{\alpha}}$
T
L_2
L_1

Example



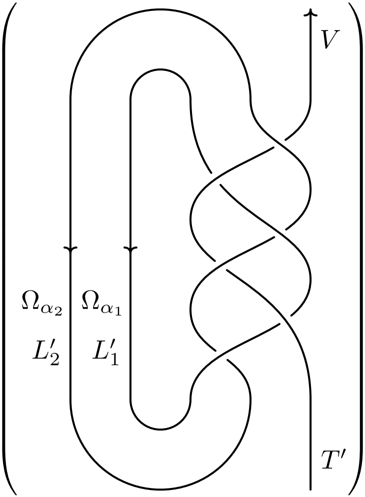
$\langle \omega, m_1 \rangle = \bar{\alpha}_1$
m_1
$\langle \omega, m_2 \rangle = \bar{\alpha}_2$
m_2
Ω_{α_2}
Ω_{α_1}
V
L_2
L_1
T

Example



Example

$\mathscr{D}^{-3}\ \mathrm{t}_V^H$



Technical Hurdle

Warning

Ω_α is defined only for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$

We need to work with special surgery presentations

Definition (Computable surgery presentation $L \subset S^3$ of (M, T, ω))

$$\langle \omega, m_i \rangle \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z} \quad \forall \; m_i$$  $$L_i \subset L$$

Renormalized Hennings Theory

Strategy

Work directly over $\bar{U}$

- Basis of $\bar{U}$ given by $\{E^a F^b K^c \mid 0 \leqslant a, b, c \leqslant r - 1\}$
- Right integral $\lambda \in \bar{U}^*$ defined by $\lambda(E^a F^b K^c) = \delta_{a,r-1}\delta_{b,r-1}\delta_{c,1}$
- $\bar{\mathrm{t}}_{\bar{U}}(f) := \lambda(f(1)K)$ for every $f \in \mathrm{End}_{\bar{\mathscr{C}}}(\bar{U})$

Sketch of Renormalized Hennings Construction

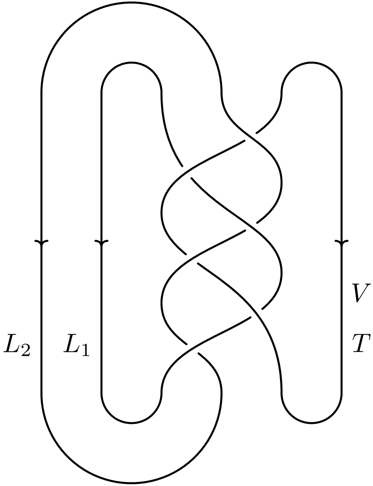
- (M, T) admissible connected pair
- $L = L_1 \cup \ldots \cup L_\ell \subset S^3$ oriented surgery presentation of M of signature $\sigma(L)$
- $L_{\bar{U}}$ obtained from L by labeling every component with $\bar{U}$
- $L'_{\bar{U}}$ obtained from $L_{\bar{U}}$ by cutting every component like a positive string link
- T' obtained from T by cutting an edge of color $V \in \mathrm{Proj}(\bar{\mathscr{C}})$

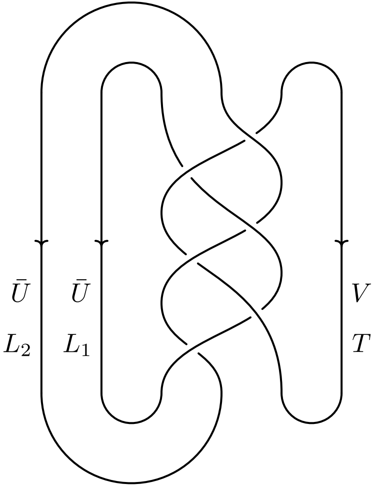
Renormalized Hennings Invariant

$$\mathrm{H}'_{\bar{U}}(M,T) := \mathscr{D}^{-1-\ell}\delta^{-\sigma(L)}\bar{\mathrm{t}}_V((\lambda^{\otimes \ell} \otimes \mathrm{id}_V) \circ F_{\bar{\mathscr{C}}}(L'_{\bar{U}} \cup T') \circ (1^{\otimes \ell} \otimes \mathrm{id}_V))$$

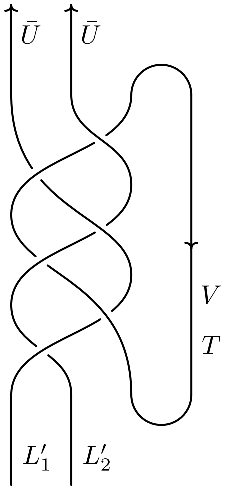
$$\mathscr{D} := \frac{\{1\}^{r-1}}{\sqrt{r}[r-1]!} \qquad \delta := i^{-\frac{r-1}{2}} q^{\frac{r-3}{2}}$$

Example

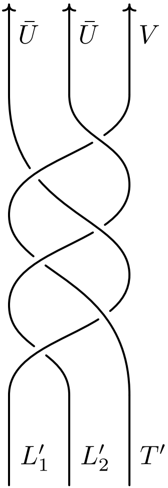



$\bar{U}$
L_2
$\bar{U}$
L_1
V
T

Example

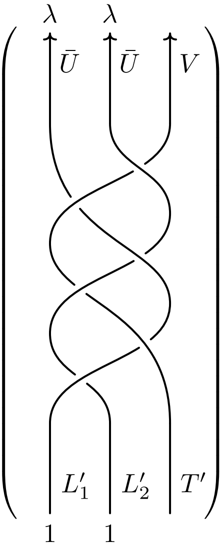


Example



Example

$\mathscr{D}^{-3}\ \bar{\mathrm{t}}_V$



Thank You for Your Attention

ご清聴ありがとうございました