Strongly quasipositive links, cyclic branched covers and L-spaces

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L-spaces

Heegaard Floer theory is a homology package of 3-manifold invariants developped by Ozsváth and Szabó which is relatively powerful in distinguishing manifolds from each other.

The simplest version of these invariants come in the form of $\mathbb{Z}_2$-graded abelian groups $\widehat{HF(M)}$, whose. Euler characteristic verifies :

$$\chi(\widehat{HF}(M)) = |H_1(M;\mathbb{Z})|$$

.

L-spaces are the class of rational homology spheres M for which the Heegaard Floer homology is as simple as possible, which means that :

$$\dim_{\mathbb{Z}_2} \widehat{HF(M)} = |H_1(M;Z)|.$$

Exemples of L-spaces include S^3, the Lens saces, and more generally all manifolds admitting elliptic geometry.

L-spaces

The 2-fold branched covering of any alternating knots or links is an L-space (Ozsváth-Szabó), providing infinitly many examples of hyperbolic L-spaces.

Conjecture (Boyer-Gordon-Watson)

For a closed, connected, orientable, irreducible 3-manifold M the following three properties are equivalent :

1. *M is not an L-space ;*
2. *M carries a co-oriented taut foliation ;*
3. *M has a left-orderable fundamental group.*

If M carries a co-orientable taut foliation then it is not an L-space [Ozsváth- Szábo].

The conjecture is true for a graph manifolds [Hanselman-Rasmussen-Rasmussen-Watson].

L-spaces

Here is the statement of a Heegaard-Floer version of Poincaré conjecture, due to Ozsváth and Szábo.

Conjecture (O-Z)

The sphere S^3 and the Poincaré sphere are the only prime integer homology sphere L-spaces.

The conjecture holds when M is obtained by surgery on a knot in S^3.

It is widely open for n-fold cyclic covers of S^3 branched over a knot, even for $n = 2$.

Conjecture (O-Z for cyclic branched covers)

An L-space integer homology sphere is a n-fold cyclic cover of S^3 branched over a non trivial prime knot K, if and only if $n = 2, 3$ or 5 and K is the $(3, 5), (2, 5)$ or $(2, 3)$ torus knot.

L-spaces

Question

Which knots or links in S^3 have L-spaces as n-fold cyclic branched coverings ?

We will address this question for the class of strongly quasipositive links.

A link L is *quasipositive* (**qp**) if there is a smooth holomorphic curve $C \subset \mathbb{C}^2$ which is transverse to $S^3 = \partial B^4$ such that $L = C \cap S^3$.

L is *strongly quasipositive (***sqp***)* if there is a smooth holomorphic curve C as above such that $L = C \cap S^3$ and $F = C \cap B^4$ can be isotoped (rel L) into S^3.

A link is *positive* if it has a diagram all of whose crossings are positive.

Thm (Lee Rudolph, Takuji Nakamura)

$\{$*positive links* $\} \subsetneq \{$*sqp links*$\} \subsetneq \{$ *qp links* $\}$.

L-space knots

L-space knots are knots producing L-spaces by Dehn surgery.

They are **sqp** and fibred [O-Z, Y. Ni, M. Hedden]. Torus knots are L-space knots.

Question (Allison Moore)

If K is a hyperbolic L-space knot, is it true that $\Sigma_2(K)$ is not an L-space ?

Here is a generalisation of A. Moore's Question.

Conjecture

If K is a prime fibred **sqp** *prime for which some $\Sigma_n(K)$ is an L-space, then K is a $(2, k)$, $(3, 4)$, or $(3, 5)$ torus knot.*

This can be shown to be true for example for :

- prime fibred alternating, or Montesinos, or special arborescent, **sqp** knots,

- positive closed braids (by S. Baader)

Genera

All links $L \subset S^3$ are oriented and considered up to mirror image.

Convention : L is a **sqp** or a **qp** link if either L or its mirror image has this property.

A *Seifert surface* for L is an oriented surface with no closed components whose oriented boundary is L.

For a link $L \subset S^3$ we associate 3 different genera :

-The Seifert genus $g(L) =$ the minimal genus of a Seifert surface for L.

-The slice genus $g_4(L) =$ the minimal genus of a smooth properly embedded surface bounding L in B^4.

-The (topologically) locally flat 4-ball genus $g_4^{top}(L) =$ the minimal genus of a locally flat properly embedded surface bounding L in B^4.

The following inequalities hold : $g_4^{top}(L) \leq g_4(L) \leq g(L)$

Quasipositivity

Thm (Kronheimer-Mrowka)

Let $L = C \cap S^3$ be **qp** *with $F = C \cap B^4$ a piece of holomorphic curve :*

$$\chi(F) = \max\{\chi(F') : F' \text{ a smooth slice surface for } L\} = \chi_4(L)$$

Hence if L is a knot K, $g_4(K) = g(F)$

Moreover if L is an **sqp** *link with F isotopic (rel L) into S^3, then :*

$$\chi_4(K) = \chi(F) = \chi(L).$$

Seifert surface

Each Seifert surface F of L determines a bilinear Seifert form

$\mathcal{S}_F : H_1(F) \times H_1(F) \to \mathbb{Z}$ with intersection form $\mathcal{S}_F - \mathcal{S}_F^T$ on $H_1(F)$.

The *Alexander polynomial of L* is the element $\Delta_L(t)$ of $\mathbb{Z}[t, t^{-1}]$, represented by $\det(\mathcal{S}_F - t\mathcal{S}_F^T)$, up to multiplication by units $\pm t^k$.

$\forall\, \zeta \in S^1$, $\mathcal{S}_F(\zeta) = (1-\zeta)\mathcal{S}_F + (1-\bar{\zeta})\mathcal{S}_F^T$ is a Hermitian form on $H_1(F)$ whose signature and nullity are independent of F.

The *Tristram-Levine signature function of L* is defined by :

$\sigma_L : S^1 \to \mathbb{Z},\ \sigma_L(\zeta) = \text{signature}(\mathcal{S}_F(\zeta))$,

while the *nullity function of L* is defined by :

$\eta_L : S^1 \to \mathbb{Z},\ \eta_L(\zeta) = \text{nullity}(\mathcal{S}_F(\zeta))$.

Signature function

Here is a list of some well-known properties of σ_L, η_L, and Δ_L.

0- $\sigma_L(-1)$ is the classical Murasugi signature, denoted by $\sigma(L)$;

1- $\sigma_L(\zeta) = \sigma_L(\bar{\zeta})$ and $\eta_L(\zeta) = \eta_L(\bar{\zeta})$ for all ζ ;

2- σ_L and η_L are constant on the components of $S^1 \setminus \Delta_L^{-1}(0)$;

3- $\eta_L(\zeta) \leq m - 1$ for $\zeta \in S^1 \setminus \Delta_L^{-1}(0)$;

Let $\Sigma_n(L)$ be the n-fold cyclic branched covering of the oriented link L.

4- $|H_1(\Sigma_n(L))| = \prod_{j=1}^{n-1} |\Delta_L(\zeta_n^j)|$ with $\zeta_n = \exp(\frac{2\pi i}{n})$.

5- $\beta_1(\Sigma_n(L)) = \sum_{j=1}^{n-1} \eta_L(\zeta_n^j)$,

$\Rightarrow \beta_1(\Sigma_n(L)) \geq (n-1)(\mu - 1)$ if L has a Seifert surface F with μ components.

$\Rightarrow F$ is connected if $\Sigma_n(L)$ is a $\mathbb{Q}$-homology sphere.

Murasugi-Tristram inequality

Thm (Murasugi-Tristram Inequality)

Let $(F, \partial F) \subset (B^4, \partial B^4)$ be a locally flat, compact, oriented surface with μ components. Let $L = \partial F$ with the induced orientation and m components. If ζ is not a root of $\Delta_L(t)$, then :

$$|\sigma_L(\zeta)| + |\eta_L(\zeta) - (\mu - 1)| \leq \beta_1(F) = 2g(F) + (m - \mu)$$

When $\Sigma_n(L)$ is a $\mathbb{Q}$-homology sphere, $\eta_L(\zeta_n^j) = 0$ for $1 \leq j \leq n-1$.

Corollary

If $H_1(\Sigma_n(L); \mathbb{Q}) = \{0\}$ and $(F, \partial F) \subset (B^4, \partial B^4)$ is as above with oriented boundary $L = \partial F$. Then for $1 \leq j \leq n-1$,

$$|\sigma_L(\zeta_n^j)| \leq 1 - \chi_4^{top}(F)$$

SQP links with L-spaces branched cyclic covers

Thm (B-Boyer-Gordon)

Let L be a **sqp** *link of m components such that $\Sigma_n(L)$ is an L-space for some $n \geq 2$. Then :*

(1) *The roots of $\Delta_L(t)$ are contained in the open subarc $]\bar{\zeta_n}, \zeta_n[\subset S^1$ containing $+1$.*

(2) $|\sigma_L(\zeta)| = 1 - \chi(L) = 2g(L) + (m-1) = \deg(\Delta_L(t))$ *for $\zeta \in$ subarc $[\zeta_n, \bar{\zeta_n}] \subset S^1$ containing -1.*

(3) $g_4^{top}(L) = g(L)$.

(4) *If $\Delta_L(t)$ is not an integer multiple of $(t-1)^{2g(L)+(m-1)}$, $\exists$ $n_3(L)$ determined by σ_L and Δ_L such that $n \leq n_3(L)$.*

SQP Links with monic Alexander polynomials

When the reduced Alexander polynomial $\Delta_L(t)$ is monic (e.g. L is fibred) we get more precise restrictions :

Thm (B-Boyer-Gordon)

Suppose that L be a **sqp** *link of m components with monic reduced Alexander polynomial $\Delta_L(t)$ which is not a power of $t-1$.*

(1) $\Sigma_n(K)$ *is not an L-space for* $n \geq 6$.

(2) *If* $\Sigma_n(K)$ *is an L-space for* $2 \leq n \leq 5$, *then* $|\sigma(L)| = 2g(L) + (m-1)$ *and* $\Delta_L(t)$ *is a product of cyclotomic polynomials. Moreover :*

(a) $n = 3 \Longrightarrow \Delta_L(t) = \Phi_4^k \Phi_5^m \Phi_6^p \Phi_{10}^q$;

(b) $n = 4 \Longrightarrow \Delta_L(t) = \Phi_5^p \Phi_6^q$;

(c) $n = 5 \Longrightarrow \Delta_L(t) = \Phi_6^p$.

SQP Knots with monic Alexander polynomials

For the case of knots we get the following restrictions :

Corollary

Suppose that K is a **sqp** *knot with monic Alexander polynomial.*

(1) $\Sigma_n(K)$ *is not an L-space for* $n \geq 6$.

(2) *If* $\Sigma_n(K)$ *is an L-space for* $2 \leq n \leq 5$, *then* $|\sigma(K)| = 2g(K)$ *and* $\Delta_K(t)$ *is a product of cyclotomic polynomials. Moreover :*

(a) $n = 3 \Longrightarrow \Delta_K(t) = \Phi_6^n \Phi_{10}^m$;

(b) $n \in \{4, 5\} \Longrightarrow \Delta_K(t) = \Phi_6^n$.

SQP knots with monic Alexander polynomials

The result is sharp : a torus knot K is fibred and **sqp**.

Moreover $\Sigma_n(K)$ is an L-space if and only if :

$\mathbf{n = 2}$ and K is the $(2, k)$, $(3, 4)$, or $(3, 5)$ torus knot.
In each case, $\Delta_K(t)$ is a non-trivial product of cyclotomics ;

$\mathbf{n = 3}$ and K is a $(2, 3)$ or $(2, 5)$ torus knot.
In the first case, $\Delta_K(t) = \Phi_6$ while in the the second case, $\Delta_K(t) = \Phi_{10}(t)$;

$\mathbf{n = 5}$ and K is a $(2, 3)$ torus knot. In this case, $\Delta_K(t) = \Phi_6$.

Filip Misev constructed an infinite family of hyperbolic, fibred, **sqp** knots with Alexander polynomial Φ_{10} and **maximal** signature.

L-space knots with L-space branched covers

If K is an L-space knot, the known restrictions on Δ_K imply that it is either Φ_6 or Φ_{10} when $n \in \{3, 4, 5\}$.

Corollary

If K is an L-space knot such that $\Sigma_n(K)$ is an L-space, then $n \leq 5$. Moreover,

(1) *if $n = 4, 5 \implies \Delta_K(t) = \Phi_6$ and K is the $(2, 3)$ torus knot*

(2) *if $n = 3 \implies K$ is either the $(2, 3)$ or $\Delta_K(t) = \Phi_{10}(t)$.*

We expect K to be the $(2, 3)$ or $(2, 5)$ torus knot in the case $n = 3$.

If $\Delta_K(t) = \Phi_{10}(t)$, then $\Sigma_3(K)$ is a $\mathbb{Z}$-homology 3-sphere.

$\Sigma_3(K)$ L-space would imply that K is the $(2, 5)$ torus knot if the O-S conjecture is true.

Sketch of proof for K sqp knot

$K = C \cap \partial B^4$, C smooth holomorphic curve transverse to $\partial B^4 = S^3$

$F = C \cap B^4$ and $\Sigma_n(F) \to B^4$ the n-fold cyclic cover branched over F

$\Sigma_n(F)$ is a Stein 4-manifold with strictly pseudo-convex boundary $\Sigma_n(K)$

Assume $\Sigma_n(K) = \partial \Sigma_n(F)$ is an L-space.

$H_1(\Sigma_n(K), \mathbb{Q}) = \{0\} \Rightarrow (H_2(\Sigma_n(F); \mathbb{C}), \cdot)$ non-singular intersection form.

$\Sigma_n(K)$ an L-space $\Rightarrow (H_2(\Sigma_n(F); \mathbb{C}), \cdot)$ negative definite [Ozsváth-Szabó]

$\Rightarrow 2g(K) \geq 2g_4^{top}(K) \geq |\sigma(\zeta_n)| = \beta_2(\Sigma_n(F)) \geq 2g(K)$.

So $2g(K) = 2g_4^{top}(K) = |\sigma(\zeta_n)|$.

Sketch of proof for K sqp knot

Degree$(\Delta_K) \leq 2g(K) = |\sigma(\zeta_n)| \Rightarrow \Delta_K^{-1}(0) \subset$ subarc $]\bar{\zeta_n}, \zeta_n[\subset S^1$ containing 1 .

$\Rightarrow |\sigma(\zeta)| = 2g(K)$ if $\zeta \in$ the closed subarc $[\zeta_n, \bar{\zeta_n}] \subset S^1$ containing -1.

Let $n_3(K)$ be the largest integer m such that $\Delta_K^{-1}(0) \subset]\bar{\zeta_m}, \zeta_m[\subset S^1$.

Then $n \leq n_3(K)$.

If Δ_K is monic, Kroneckers thm $\Rightarrow \Delta_K$ product of cyclotomic polynomials.

For $n \geq 6$ and $a \geq 2$, the cyclotomic polynomial Φ_a has a root in $[\zeta_n, \bar{\zeta_n}] \subset S^1$.

Thus, $n \leq 5$.

Case-by-case analysis when $n = 3, 4, 5$, yields the listed restrictions on Δ_K.

Satellite knots

Next we consider **sqp** satellite knots.

Proposition

Let K be a **sqp** *satellite knot with non-trivial companion C and pattern P of winding number w.*

Let K_1 be the knot whose exterior is obtained from that of K by pinching the exterior of C to a solid torus.

If $\Sigma_n(K)$ is an L-space for some $n \geq 2$ then $|w| = 0, 1$. Moreover :

(1) *If $|w| = 1$, then $|\sigma(C)| = 2g(C)$ and $|\sigma(K_1)| = 2g(K_1)$.*

(2) *If $|w| = 0$, then $g(K_1) = g(K)$.*

Case (2) does not occur when K is a fibred **sqp** satellite knot.

Proof

$g(K) \geq |w| g(C) + g(K_1)$ (H. Schubert)

- $\sigma(K) = \sigma(C) + \sigma(K_1)$ if $|w|$ is odd ;
- $\sigma(K) = \sigma(K_1)$ if $|w|$ is even (Y. Shinohara).

K is **sqp** and $\Sigma_n(K)$ is an L-space for some $n \geq 2 \Rightarrow$

$|\sigma(K)| = 2g(K) \geq 2|w|g(C) + 2g(K_1) \geq |w||\sigma(C)| + |\sigma(K_1)| \geq |\sigma(K)|$

$\Rightarrow$ this sequence of inequalities is a sequence of equalities.

$w \neq 0 \Rightarrow |w| = 1$, $|\sigma(C)| = 2g(C)$ and $|\sigma(K_1)| = 2g(K_1)$

$w = 0 \Rightarrow 2g(K_1) \leq 2g(K) = |\sigma(K)| = |\sigma(K_1)| \leq 2g(K_1)$

$\Rightarrow g(K_1) = g(K)$.

Satellite knots

K. Baker and K. Motegi showed that satellite L-space knots can be expressed as a satellite knot where the pattern is a braid

Corollary

For a satellite L-space knot $\Sigma_n(K)$ is never an L-space for $n \geq 2$.

The following conjecture would imply that the only L-space knots for which some $\Sigma_n(K)$ can be an L-space are iterated torus knots.

Conjecture (E. Li and Y. Ni)

If K is an L-space knot and each root of its Alexander polynomial lies on the unit circle, then K is an iterated torus knot.

In this case K must be a torus knot by the corollary and thus a $(2, k)$, $(3, 4)$ or $(3, 5)$ torus knot.

Simply laced arborescent links

Conjecture

If L is a prime, fibred, strongly quasipositive link for which some $\Sigma_n(L)$ is an L-space, then L is simply laced arborescent.

The boundary $L(\Gamma)$ of the plumbing of positive Hopf bands according to one of the trees $\Gamma = A_m (m \geq 1)$, $D_m (m \geq 4)$, E_6, E_7, E_8 is called *simply laced arborescent* :

(i) $L(A_m) = T(2, m+1)$
(ii) $L(D_m) = P(-2, 2, m-2)$
(iii) $L(E_6) = P(-2, 3, 3) = T(3, 4)$
(iv) $L(E_7) = P(-2, 3, 4)$
(v) $L(E_8) = P(-2, 3, 5) = T(3, 5)$

$T(p, q)$ is the (p, q) torus link and $P(p, q, r)$ the (p, q, r) pretzel link.

For such a link L, $\pi_1(\Sigma_2(L))$ is finite and so $\Sigma_2(L)$ is a L-space.

Quasipositive braids

Thm (Rudolph, B-Orevkov)

A link $L \subset S^3$ is :

qp $\iff L = \hat{\beta}$ *for some* $\beta = \prod_{i=1}^{k} w_{i\sigma_j(i)} w_i^{-1} \in B_n,\ n \geq 1$

sqp $\iff L = \hat{\beta}$ *for some* $\beta = \prod_{i=1}^{k} w_{i\sigma_j(i)} w_i^{-1} \in B_n,\ n \geq 1,$

where $w_i = \sigma_p \sigma_{p+1} \cdots \sigma_{j(i)-1}, p < j(i)$.

Corollary

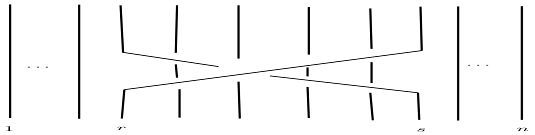
If $L = \hat{\beta}$ *is* **qp**, *then* $\chi_4(L) = n - k$

If $L = \hat{\beta}$ *is* **sqp**, *then* $\chi(L) = n - k = \chi_4(L)$

BKL-braids

Birman-Ko-Lee introduced a presentation for the braid group B_n with generators the strongly quasipositive braids a_{rs}, $1 \leq r < s \leq n$, given by

$$a_{rs} = (\sigma_r \sigma_{r+1} \dots \sigma_{s-2}) \sigma_{s-1} (\sigma_r \sigma_{r+1} \dots \sigma_{s-2})^{-1} \quad (1)$$



A braid in B_n is called *BKL-positive* if it can be expressed as a word in positive powers of the generators a_{rs}.

BKL- positive braids

BKL-positive elements in B_n coincide with strongly quasipositive n-braids.

The *dual Garside element* $\delta_n = \sigma_1\sigma_2\ldots\sigma_{n-1} \in B_n$ plays an important role.

We call the *BKL-exponent* of a strongly quasipositive link L the integer

$$k(L) = \max\{k : L = \widehat{\delta_n^k P},\ n \geq 2,\ k \geq 0 \text{ and } P \in B_n \text{ is BKL-positive}\}$$

One shows that $k(L) < \infty$. Moreover $k(L) \geq 2$ when L is simply laced arborescent.

Thm (B-Boyer-Gordon)

Let L be a prime **sqp** *link with BKL-exponent $k(L) \geq 2$. Then $\Sigma_n(L)$ is an L-space for some $n \geq 2$ if and only if L is simply laced arborescent.*

BKL- positive braids

A **sqp** link L with $k(L) \geq 1$ is fibred [Banfield]

A link L of m components is said to be *definite* if the signature is maximal :

$$|\sigma(L)| = 2g(L) + (m-1)$$

.

A **sqp** link L for which some $\Sigma_n(L)$ is an L-space is definite,

Hence the Theorem follows from the following characterisation of simply laced arborescent links :

Thm (B-Boyer-Gordon)

Let L be a prime **sqp** *link. Then L is simply laced arborescent if and only if it is definite and $k(L) \geq 2$.*

BKL- positive braids

The condition $k(L) \geq 2$ cannot be relaxed : there are prime **sqp** definite links with $k(L) = 1$.

The simply laced arborescent links are all definite positive braid links.

A key ingredient for the proof is the following result :

Thm (Baader)

A prime positive braid link is simply laced arborescent if and only if it is definite.

Baader's theorem reduces the proof to the following result :

Thm (B-Boyer-Gordon)

If the closure of a BKL-positive word $\delta_n^2 P \in B_n$, $n \geq 3$, is a definite link, then $\delta_n^2 P$ is conjugate to a positive braid.